ISRAEL JOURNAL OF MATHEMATICS 86 (1994), 61–106

MORERA AND MEAN-VALUE TYPE THEOREMS IN THE HYPERBOLIC DISK

BY

CARLOS BERENSTEIN*

Mathematics Department and Systems Research Center
University of Maryland, College Park, MD 20742, USA

AND

DANIEL PASCUAS**

Departament de Matemàtica Aplicada i Anàlisi
Facultat de Matemàtiques, Universitat de Barcelona, 08071 Barcelona, Spain

ABSTRACT

Some Morera and mean-value type theorems are proved in the hyperbolic disk.

1. Statement of problems and results

1.1 In [BZ2] a general Pompeiu transform was introduced as follows. Let X be a locally compact topological space, G a topological group acting continuously and transitively on X, and μ a Radon measure on X which is invariant under G. For a fixed compact $K \subset X$ one defines the **Pompeiu transform** $P = P_K$ by

$$P: C(X) \longrightarrow C(G)$$
$$Pf(\sigma) := \int_{\sigma(K)} f\, d\mu.$$

* Partially supported by NSF grants DMS-9000619 and CDR-8803012.
** Partially supported by NSF grants DMS-8703072 and DMS-9000619, and DGICYT grant PB 89-0311.
Received December 20, 1992 and in revised form March 31, 1993

Note that the invariance of μ implies that $Pf(\sigma) = \int_K (f \circ \sigma)\, d\mu$. The problem posed is to decide whether K has the **Pompeiu property**, i.e., whether P is injective.

A typical example is that of $X = \mathbb{C}$, $G = M(2)$, the group of holomorphic rigid motions of the plane, $\mu = m$, the Lebesgue measure, and K, the closure of a Jordan domain Ω.

Assuming that Ω is a Lipschitz domain (in particular, $\partial\Omega$ has no cusps), we can also consider whether $\Gamma = \partial\Omega$ has the (global) **Morera property**, i.e., whether any $f \in C(\mathbb{C})$ that satisfies

$$\int_{\sigma(\Gamma)} f(z)\, dz = 0, \qquad \text{for every } \sigma \in G,$$

is necessarily an entire function. It is easy to see that it is enough to consider functions $f \in C^1(\mathbb{C})$. Then, by Stokes' theorem, the above condition is equivalent to

$$\int_{\sigma(K)} \frac{\partial f}{\partial \bar z}\, d\bar z \wedge dz = 2i \int_{\sigma(K)} \frac{\partial f}{\partial \bar z}\, dm = 0, \qquad \text{for every } \sigma \in G.$$

So that, in this situation, the Morera and Pompeiu properties are equivalent. These problems have led to a considerable amount of research, and we refer the reader to [Z] for an account of it. For instance, one can prove that if Γ is a triangle (or any Lipschitz Jordan curve that is not real analytic) then it has the Morera property.

A priori, it would seem that the set up we have just stated is the only reasonable one to obtain interesting results, but it turns out that there are natural questions where it is not true that the whole group G operates on X, only a neighborhood of the identity, or the measure μ cannot be taken to be invariant. For instance, in [BG] the local Morera problem was considered and it was shown that, given a Jordan curve Γ with the global Morera property and such that it is contained in the open disk $D(0, 1/2)$ of center 0 and radius $1/2$, then any function $f \in C(D(0,1))$ which satisfies

$$\int_{\sigma(\Gamma)} f(z)\, dz = 0, \qquad \text{for every } \sigma \in M(2),\ \sigma(\Gamma) \subset D(0,1),$$

is holomorphic on $D(0,1)$. Clearly, only a neighborhood of the identity of $M(2)$ operates on $X = D(0,1)$. On the other hand, as explained above, we can still couch this problem in terms of the Pompeiu problem with an invariant measure.

In this paper we consider several instances where the measure cannot be taken invariant.

1.2 Let $\mathbb{D}$ be the unit disk in the complex plane, and let $\mathcal{M}$ be the Möbius group of conformal automorphisms of $\mathbb{D}$.

Let Γ be a piecewise-C^1 Jordan curve in $\mathbb{D}$. If f is a holomorphic function on $\mathbb{D}$ then, for every $\sigma \in \mathcal{M}$, $f \circ \sigma$ is holomorphic too, and by Cauchy's theorem we have

$$\int_\Gamma (f \circ \sigma)(z)\, dz = 0, \qquad \text{for every } \sigma \in \mathcal{M}. \tag{1.1}$$

Our first goal is to investigate the converse result or Morera problem:

When is it true that every $f \in C(\mathbb{D})$ satisfying (1.1) is necessarily holomorphic on $\mathbb{D}$?

Observe that the measure $dz_{/\Gamma}$ is not invariant under the action of $\mathcal{M}$, and there is no way to formulate this problem using invariant measures.

We denote, as usual, $D(c, R)$ the open Euclidean disk in $\mathbb{C}$ centered at $c \in \mathbb{C}$ with radius $R > 0$. If Γ is a circle, i.e., $\Gamma = \partial D(c, R)$, the above problem is completely solved by means the following circular Morera theorem:

THEOREM 1: *Let $D = D(c, R) \subset\subset \mathbb{D}$. Assume that a function $f \in C(\mathbb{D})$ satisfies*

$$\int_{\partial D} (f \circ \sigma)(z)\, dz = 0, \qquad \textit{for every } \sigma \in \mathcal{M}. \tag{1.2}$$

(a) *If $c \neq 0$ then f is holomorphic on $\mathbb{D}$.*

(b) *If $c = 0$ there are nonconstant radial real analytic (so nonholomorphic) functions on $\mathbb{D}$ satisfying (1.2). But if $f \in C(\mathbb{D})$ verifies (1.2) for a family of circles $\partial D(0, R_j)$, $j \in J$, such that the equations*

$$P_z^{-1}\left(\frac{1 + R_j^2}{1 - R_j^2}\right) = 0 \qquad (j \in J) \tag{1.3}$$

have no common solution $z \in \mathbb{C}$, then f is holomorphic on $\mathbb{D}$.

Here we denote P_z^ν, as usual, the associated Legendre functions of the first kind.

Remarks: (a) The result of part (b) is sharp in the sense that if the radii r_j, $j \in J$, do not satisfy the stated condition then there are nonconstant radial real analytic (so nonholomorphic) functions verifying (1.2) for the circles $\partial D(0, R_j)$, $j \in J$.

(b) When $c = 0$ only two radii R_1, R_2 are sufficient to imply the holomorphy of f. In fact, using the classical theory of selfadjoint ordinary differential operators (see [LS]) it is possible to show that the family of pairs of radii R_1, R_2 failing the condition stated in part b) above has measure zero in the unit square.

For general curves Γ we obtain the following "general" Morera theorem:

THEOREM 2: *Let $\Omega \subset\subset \mathbb{D}$ be a Jordan domain of class $C^{2,\varepsilon}$, for some $0 < \varepsilon < 1$. Suppose that the Jordan curve $\Gamma = \partial\Omega$ is not real analytic. Assume $f \in C(\mathbb{D})$ satisfies*

$$\int_\Gamma (f \circ \sigma)(z)\, dz = 0, \qquad \text{for every } \sigma \in \mathcal{M}. \tag{1.4}$$

Then f is holomorphic on $\mathbb{D}$.

In order to compare the above results with some previous ones, we recall the invariant version of the Morera problem we have posed. It consists of deciding whether every $f \in C(\mathbb{D})$ which verifies

$$\int_{\sigma(\Gamma)} f(z)\, dz = 0, \qquad \text{for every } \sigma \in \mathcal{M}, \tag{1.5}$$

is holomorphic on $\mathbb{D}$.

An argument similar to the one we made in 1.1 for the Euclidean case shows that the above problem is equivalent to the Pompeiu problem in the hyperbolic disk. (Here the Radon measure involved is the hyperbolic invariant measure $d\mu(z) = dm(z)/(1 - |z|^2)^2$.)

When Γ is a circle the invariant problem was completely solved by the following result due to Berenstein and Zalcman:

THEOREM 3 ([BZ1, p. 125–6]):

(a) *Let $D = D(0, R)$, $0 < R < 1$. Then there are nonconstant radial real analytic (so nonholomorphic) functions on $\mathbb{D}$ satisfying (1.5) with $\Gamma = \partial D$.*

(b) *If $f \in C(\mathbb{D})$ verifies (1.5) for Γ running on a family of circles $\partial D(0, R_j)$, $j \in J$, such that the equations (1.3) have no common solution $z \in \mathbb{C}$, then f is holomorphic on $\mathbb{D}$.*

The remarks to Theorem 1 are also valid for Theorem 3.

Note that in Theorem 1, in case $c \neq 0$, the hypothesis always implies the holomorphy of the function. In other words, there are no exceptional radii and we have a one-radius theorem, in contrast with Theorem 3 and the case $c = 0$ of Theorem 1, both of which are two-radii theorems. Observe also that the hypothesis of the noninvariant theorem for $c = 0$ is in some sense close to that of the invariant theorem, although they are formally different. Furthermore, both results have the same critical set of radii.

For general curves, the best known result for the above invariant problem, which was obtained by Berenstein and Shahshahani, is much better than ours:

THEOREM 4 ([BS, p. 125-6]): *Let $\Omega \subset\subset \mathbb{D}$ be a Lipschitz Jordan domain, whose boundary $\Gamma = \partial\Omega$ is not a real analytic curve. Then, if $f \in C(\mathbb{D})$ satisfies* (1.5), *it is holomorphic on $\mathbb{D}$.*

The reason that we cannot prove Theorem 2 for nonreal-analytic boundaries of Lipschitz Jordan domains is rather technical. It depends on the nonavailability of a strong enough regularity theorem for elliptic systems of differential equations with boundary conditions. In the invariant case, only an elliptic differential equation with boundary conditions is involved and there is a deep regularity result for this type of equations, due to Caffarelli [C], which allows one to prove Theorem 4.

For a more detailed exposition on Morera problems and related topics we recommend the recent survey [BCPZ]. Other holomorphy tests which are based on the Cauchy integral formula (instead of Cauchy's theorem) can be found in [CP].

1.3 We also consider mean-value type problems of the same noninvariant nature as the above Morera problem.

Let $D = D(c, R) \subset\subset \mathbb{D}$. If f is an harmonic function on $\mathbb{D}$ then, for every $\sigma \in \mathcal{M}$, $f \circ \sigma$ is harmonic too, and therefore it has the mean-value property

$$(1.6) \qquad \int_{\partial D} (f \circ \sigma)(\zeta) \frac{|d\zeta|}{2\pi R} = (f \circ \sigma)(c), \qquad \text{for every } \sigma \in \mathcal{M}.$$

Thus it turns out that the natural mean-value problem here is to study the converse result: Is it true that every $f \in C(\mathbb{D})$ satisfying (1.6) is necessarily harmonic on $\mathbb{D}$?

Observe that when $c = 0$ the above problem can be stated using an invariant measure, namely the hyperbolic arc-length measure. In fact, if $c = 0$, letting $R = \tanh(r/2)$, for $r > 0$, it is easy to check that condition (1.6) can be rewritten as

$$(1.7) \quad \frac{1}{\pi \sinh r} \int_{\sigma(\partial D(0,R))} f(\zeta) \frac{|d\zeta|}{1 - |\zeta|^2} = (f \circ \sigma)(0), \qquad \text{for every } \sigma \in \mathcal{M}.$$

The solution to the above invariant problem was given by the following two-radii theorem:

THEOREM 5 ([BZ1, Thm. 3]):

(a) *Let* $D = D(0, R)$, $0 < R < 1$. *Then there are nonconstant radial real analytic (so nonharmonic) functions on* $\mathbb{D}$ *satisfying* (1.6).

(b) *If* $f \in C(\mathbb{D})$ *verifies* (1.6) *for a family of circles* $\partial D(0, R_j)$, $j \in J$, *such that the equations*

$$P_z\left(\frac{1+R_j^2}{1-R_j^2}\right) = 1 \qquad (j \in J)$$

have no common solution $z \in \mathbb{C}$, *except* $z = 0, 1$, *then* f *is harmonic on* $\mathbb{D}$.

Here P_z denotes, as usual, the Legendre function of the first kind P_z^0.

The remarks to Theorem 1 also hold for the above result (replacing holomorphic by harmonic).

On the other side, the mean-value problem we have stated can not be formulated using an invariant measure if $c \neq 0$. In that case we have the following one-radius mean-value theorem:

THEOREM 6: *Let* $D = D(c, R) \subset\subset \mathbb{D}$ *and* $c \neq 0$. *Every function* $f \in C(\mathbb{D})$ *satisfying* (1.6) *is harmonic on* $\mathbb{D}$.

Observe the similarity of the above result with Theorem 1.a). We can say that the first one is the harmonic version of the second one.

1.4 Finally, we consider a noncentered or general mean-value problem.

Let $\mathcal{P} = \mathcal{P}_D$ be the Poisson kernel of the disk $D = D(c, R) \subset\subset \mathbb{D}$, that is

$$\mathcal{P}_D(z, \zeta) = \frac{R^2 - |z-c|^2}{|\zeta - z|^2} \qquad (z \in D,\ \zeta \in \partial D).$$

Fix a point $a \in D$. If f is an harmonic function on $\mathbb{D}$ then so is $f \circ \sigma$, for every $\sigma \in \mathcal{M}$, and therefore

$$\int_{\partial D} \mathcal{P}(a, \zeta)(f \circ \sigma)(\zeta)\, \frac{|d\zeta|}{2\pi R} = (f \circ \sigma)(a), \qquad \text{for every } \sigma \in \mathcal{M}. \tag{1.8}$$

Thus the general mean-value problem is to decide whether every function $f \in C(\mathbb{D})$ satisfying (1.8) has to be harmonic on $\mathbb{D}$.

Note that, when the point a coincides with the hyperbolic center of D, i.e., $D(c, R) = \tau(D(0, r))$, where τ is the conformal automorphism

$$\tau(z) = \frac{z + a}{1 + \overline{a}z} \qquad (z \in \mathbb{D}),$$

by making the change of variable $\zeta = \tau(\eta)$ the identity (1.8) can be written as

$$\int_{\partial D(0,r)} (f \circ \sigma)(\eta) \frac{|d\eta|}{2\pi r} = (f \circ \sigma)(0), \qquad \text{for every } \sigma \in \mathcal{M}.$$

Thus in that case the general mean-value problem is identical to the mean-value problem at the origin which we discussed in 1.3 and whose solution is given by Theorem 5.

For the other case we obtain the following one-radius general mean-value theorem:

THEOREM 7: *Let $D = D(c, r) \subset\subset \mathbb{D}$ and let $a \in D$. Assume that a does not coincide with the hyperbolic center of D. Then every function $f \in C(\mathbb{D})$ which satisfies (1.8) is harmonic on $\mathbb{D}$.*

Observe that Theorem 6 is just the particular case $a = c$ of Theorem 7. So we will only prove the second theorem and obtain the first one as a corollary.

1.5 The plan of the paper is the following. In the next section the basic notation and simple technical results required to prove the above stated results are introduced. Our circular and general Morera theorems (Theorems 1 and 2) are proved in Sections 3 and 4, respectively, while the proof of the general mean-value theorem (Theorem 7) is carried out in Section 5.

The proof of the circular Morera theorem as well as the general mean value theorem is done, roughly speaking, in the following way. First we show that the circular Morera problem is really the problem of proving our function is harmonic. Then the hypothesis is written as a corresponding convolution equation in the Möbius group $\mathcal{M}$. We associate to that equation a closed ideal of bi-invariant compactly supported distributions on $\mathcal{M}$, that is, radial compactly supported distributions on $\mathbb{D}$. Some harmonic analysis on $\mathcal{M}$ (basically, the spherical Fourier transform on $\mathcal{M}$ and the spectral synthesis theorem of Schwartz) shows that this ideal is either the whole space of bi-invariant compactly supported distributions on $\mathcal{M}$ and then our function is harmonic, or otherwise there is some nonharmonic function satisfying the hypothesis.

The approach in the proof of the general Morera theorem is really different, due to the impossibility of carrying out explicit computations (as in the preceding case of circles) for a general curve. After reducing the problem to an harmonic one as in the circular case, we use a *reductio ad absurdum* argument. Roughly speaking, since now the technicalities are much more complicated than in the circular case, we prove that, if a function satisfying the hypothesis is not harmonic, a certain elliptic system of differential equations with boundary conditions in the Jordan domain we considered has solutions with an adequate regularity. The use of a regularity result for such a type of systems shows the solutions are really real analytic, and then so is our curve, which is a contradiction.

We finish the paper with Section 6 , where we discuss some problems related to the mean-value ones we deal with above, but which can not be attacked with the methods we use here.

2. Some notation and technical tools

The present section is devoted to introducing some notation and technical tools we will use in the remaining sections.

Throughout this work we shall mainly follow the notation of [BZ2] and [Hel2], but, for the sake of completeness and to make easier the reading of the paper, we summarize that one we will often use in the proofs of the theorems.

2.1 We recall that the hyperbolic disk is the Riemannian manifold of constant curvature -4 obtained by endowing the unit disk $\mathbb{D}$ with its usual manifold structure and the hyperbolic inner product on its tangent bundle $T(\mathbb{D})$, which is defined to be

$$(2.1)\qquad \langle X, Y\rangle_h = \frac{\langle X, Y\rangle}{(1-|z|^2)^2} \qquad (X, Y \in T_z(\mathbb{D}),\ z\in\mathbb{D}),$$

where $\langle\cdot,\cdot\rangle$ denotes the usual Euclidean inner product. Thus the hyperbolic arc length is given by

$$(2.2)\qquad ds_h = \frac{ds}{1-|z|^2},$$

$ds = |dz|$ being the Euclidean arc length. The corresponding distance, i.e., the hyperbolic distance, is

$$d(z,w) = \frac{1}{2}\log\frac{1+\rho(z,w)}{1-\rho(z,w)} \qquad (z, w\in\mathbb{D}),$$

where ρ is the so-called pseudohyperbolic distance:

$$(2.3)\qquad \rho(z,w) = \left|\frac{z-w}{1-\overline{w}z}\right| \qquad (z,w\in\mathbb{D}).$$

The Euclidean disks which are relatively compact in $\mathbb{D}$ coincide with the hyperbolic disks (as well as the pseudohyperbolic ones). Namely, the Euclidean disk

$D = D(c, R) \subset\subset \mathbb{D}$ coincides with the pseudohyperbolic disk $\Delta(c_0, r) = \{ z \in \mathbb{D}\colon \rho(z, c_0) < r \}$, where the pseudohyperbolic center $c_0 \in \mathbb{D}$ and pseudohyperbolic radius $0 < r < 1$ are related with the corresponding Euclidean ones by the identities

$$(2.4) \qquad c = \frac{1 - r^2}{1 - r^2|c_0|^2}\, c_0 \qquad \text{and} \qquad R = \frac{1 - |c_0|^2}{1 - r^2|c_0|^2}\, r$$

(see [G, p. 3]).

The positive measure and the Laplace-Beltrami operator, i.e., hyperbolic measure and hyperbolic Laplacian, respectively, associated to the hyperbolic metric are given by

$$(2.5) \qquad d\mu(z) = \frac{1}{(1 - |z|^2)^2}\, dm(z)$$

and $\Delta_h = (1 - |z|^2)^2 \Delta$, respectively, where m and Δ are the Lebesgue measure and the Euclidean Laplacian on $\mathbb{C}$, respectively.

The hyperbolic metric is invariant under the action of $\mathcal{M}$, and so are all the above concepts derived from it.

2.2 By means of the standard identification of Möbius mappings with 2×2-matrices, the Möbius group $\mathcal{M}$ is identified with the classical Lie group

$$\mathbf{SU}(1,1) = \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} : a, b \in \mathbb{C},\ |a|^2 - |b|^2 = 1 \right\} \quad (\text{modulo } \pm Id).$$

Thus $\mathcal{M}$ is a transitive Lie transformation group of the real analytic manifold $\mathbb{D}$ (for the definition of transformation group and related concepts see [Hel1, Ch. II, §3]).

Using the usual group conventions e will denote the identity mapping on $\mathbb{D}$, and sometimes the composition $g \circ h$, for $g, h \in \mathcal{M}$, will be simply written as gh.

The subgroup $\mathcal{K}$ of $\mathcal{M}$ which leaves the origin fixed is the group of all rotations around the origin, i.e., using the above identification $\mathcal{K}$ corresponds to the compact subgroup

$$\mathbf{SO}(2) = \left\{ \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix} : a \in \mathbb{C},\ |a| = 1 \right\}$$

of $\mathbf{SU}(1,1)$. It turns out that the set $\mathcal{M}/\mathcal{K}$ of left cosets $g\mathcal{K}$ (with the quotient topology) carries a natural real analytic manifold structure such that the mapping

$$\begin{array}{ccc} \mathcal{M}/\mathcal{K} & \longrightarrow & \mathbb{D} \\ g\mathcal{K} & \longmapsto & g(0) \end{array}$$

is a real analytic diffeomorphism (see [Hel1, Ch. II, §4]). Thus $\mathbb{D}$ will be considered as the homogeneous space $\mathcal{M}/\mathcal{K}$ using the above identification. We finally recall that $(\mathcal{M}, \mathcal{K})$ is a Riemannian symmetric pair (see [Hel1, p. 209] for the definition) and, in particular, the hyperbolic disk $\mathbb{D}$ and the homogeneous space $\mathcal{M}/\mathcal{K}$ can also be identified as Riemannian globally symmetric spaces (see [Hel1, Ch. IV, §3]). In fact, an involutive automorphism on $\mathcal{M}$ satisfying the conditions of the definition of symmetric pair in this case is

$$\theta g = kgk^{-1} \qquad (g \in \mathcal{M}),$$

where $k(z) = iz$, $z \in \mathbb{D}$.

2.3 Let π denote the canonical projection (or quotient mapping) from $\mathcal{M}$ onto $\mathcal{M}/\mathcal{K} \cong \mathbb{D}$, i.e., $\pi(g) = g(0)$, for every $g \in \mathcal{M}$.

We identify locally integrable functions with distributions on $\mathbb{D}$ by means of the hyperbolic measure $d\mu$ given by (2.5), which is invariant under the action of $\mathcal{M}$. The lifting of $d\mu$ by π is a left- and right-invariant Haar measure, usually denoted by dg, on $\mathcal{M}$, i.e.,

$$\int_{\mathbb{D}} \varphi(z)\, d\mu(z) = \int_{\mathcal{M}} (\varphi \circ \pi)(g)\, dg \qquad (\varphi \in \mathcal{D}(\mathbb{D})).$$

We use that measure dg to identify locally integrable functions with distributions on $\mathcal{M}$.

We denote by dk the normalized Haar measure on $\mathcal{K}$, i.e., for $\varphi \in C(\mathcal{K})$ we have

$$\int_{\mathcal{K}} \varphi(k)\, dk = \int_0^{2\pi} \varphi(k_\theta)\, \frac{d\theta}{2\pi}, \qquad \text{where } k_\theta(z) = e^{i\theta} z.$$

Let $\delta_{\mathcal{K}}$ be the compactly supported Radon measure on $\mathcal{M}$ defined by

$$\langle \delta_{\mathcal{K}}, \varphi \rangle := \int_{\mathcal{K}} \varphi(k)\, dk \qquad (\varphi \in C(\mathcal{M})).$$

A function φ on $\mathcal{M}$ is **right-invariant** (under $\mathcal{K}$) if $\varphi(gk) = \varphi(g)$, for every $k \in \mathcal{K}$.

A distribution T on $\mathcal{M}$ is **right-invariant** (under $\mathcal{K}$) when

$$\langle T, \varphi \rangle = \langle T, g \mapsto \varphi(gk) \rangle, \qquad \text{for every } k \in \mathcal{K}.$$

Left-invariant (under $\mathcal{K}$) functions or distributions are defined in a similar way. The term "bi-invariant" means left- and right-invariant (under $\mathcal{K}$). The subscript "0" will be used to denote bi-invariance, for example, $\mathcal{E}_0'(\mathcal{M})$ is the space of compactly supported bi-invariant distributions on $\mathcal{M}$.

If φ is a right-invariant (left-invariant) function on $\mathcal{M}$ then the function $\check{\varphi}$, defined by

$$\check{\varphi}(g) := \varphi(g^{-1}) \qquad (g \in \mathcal{M}),$$

is left-invariant (respectively, right-invariant).

If T is a right-invariant (left-invariant) distribution on $\mathcal{M}$ then the distribution $\check{T}$, defined by

$$\langle \check{T}, \varphi \rangle := \langle T, \check{\varphi} \rangle \qquad (\varphi \in \mathcal{D}(\mathcal{M})),$$

is left-invariant (resp., right-invariant).

2.4 The convolution of two nice functions φ and ψ on $\mathcal{M}$ is defined by

$$(\varphi * \psi)(g) = \int_{\mathcal{M}} \varphi(gh^{-1})\psi(h)\, dh \qquad (g \in \mathcal{M}).$$

The bi-invariance of the Haar measure of $\mathcal{M}$ implies that

$$(\varphi * \psi)(g) = \int_{\mathcal{M}} \varphi(h)\psi(h^{-1}g)\, dh \qquad (g \in \mathcal{M}).$$

Then the notion of convolution is extended to distributions on $\mathcal{M}$ in the usual way.

If T and S are distributions on $\mathcal{M}$, one of them with compact support, then $T * S$ is left-invariant (right-invariant) when T (resp., S) is. Thus $\mathcal{E}_0'(\mathcal{M})$ is a topological convolution algebra.

Given any nice function φ on $\mathcal{M}$ we can consider the function φ_π on $\mathbb{D}$ which is determined by

$$\varphi_\pi \circ \pi = \varphi * \delta_K.$$

The operator $\cdot_\pi$ sends the usual spaces of functions on $\mathcal{M}$ ($L^1_{loc}(\mathcal{M})$, $\mathcal{E}(\mathcal{M})$, $\mathcal{D}(\mathcal{M}), \ldots$) onto the corresponding spaces on $\mathbb{D}$. Then any $T \in \mathcal{D}'(\mathbb{D})$ lifts to a right-invariant distribution $\widetilde{T}$ on $\mathcal{M}$ given by

$$\langle \widetilde{T}, \varphi \rangle = \langle T, \varphi_\pi \rangle \qquad (\varphi \in \mathcal{D}(\mathbb{D})).$$

If T has compact support so does $\widetilde{T}$.

The lifting of Δ_h to $\mathcal{M}$ is the differential operator $\widetilde{\Delta}_h$ on $\mathcal{M}$ defined by

$$\widetilde{\Delta}_h \varphi := (\Delta_h \varphi_\pi) \circ \pi \qquad (\varphi \in \mathcal{E}(\mathcal{M})).$$

If we consider a function φ on $\mathbb{D}$ as a distribution, then we have that

$$\widetilde{\varphi} = \varphi \circ \pi \qquad \text{and} \qquad (\widetilde{\varphi})_\pi = \varphi.$$

Moreover, any distribution T on $\mathcal{M}$ determines a distribution T_π on $\mathbb{D}$ which is defined by

$$\langle T_\pi, \varphi\rangle := \langle T, \widetilde{\varphi}\rangle \qquad (\varphi \in \mathcal{D}(\mathbb{D})).$$

The operator $\widetilde{\cdot}$ establishes a bijection from the usual spaces of distributions (functions) on $\mathbb{D}$ onto the corresponding spaces of right-invariant distributions (resp., functions) on $\mathcal{M}$, and $\cdot_\pi$ is just the inverse operator.

3. Proof of the circular Morera theorem

3.1 First we reduce our "holomorphic" problem to an "harmonic" one.

LEMMA 1: *If f is an harmonic function on $\mathbb{D}$ that verifies (1.2) then f is holomorphic on $\mathbb{D}$.*

Proof: If f is harmonic on $\mathbb{D}$ then so is $f_{\bar z} = \partial f/\partial \bar z$. Therefore it satisfies the mean-value property and so, by Stokes' theorem, we have

$$f_{\bar z}(c) = \frac{1}{\pi R^2}\int_{D(c,R)} f_{\bar z}(x+iy)\,dx\,dy = \frac{1}{i\pi R^2}\int_{\partial D(c,R)} f(z)\,dz = 0.$$

The last identity is (1.2), σ being the identity mapping on $\mathbb{D}$. But if f verifies (1.2) then so does $f\circ\sigma$, for every $\sigma \in \mathcal{M}$. Hence

$$0 = \frac{\partial(f\circ\sigma)}{\partial \bar z}(c) = f_{\bar z}(\sigma(c))\cdot\overline{\sigma'(c)},$$

so, since $\sigma'(c) \neq 0$, $f_{\bar z}(\sigma(c)) = 0$, for every $\sigma \in \mathcal{M}$.

In conclusion, $f_{\bar z} \equiv 0$, i.e., f is holomorphic on $\mathbb{D}$, because $\mathcal{M}$ acts transitively on $\mathbb{D}$.

Thus, by lemma 1, we only have to study whether (1.2) implies f is harmonic on $\mathbb{D}$.

3.2 For computational reasons that will become apparent later, it is convenient to write condition (1.2) in terms of line integrals over a circle centered at the origin. To do this we recall that the Euclidean disk $D = D(c,R) \subset\subset \mathbb{D}$ coincides with the pseudohyperbolic disk $\Delta(c_0, r)$, where c_0 and r are related with c and R by the formulas (2.4). Then the automorphism of $\mathbb{D}$

$$\sigma_0(z) = \frac{z - c_0}{1 - \overline{c}_0 z} \tag{3.1}$$

maps $\partial D = \partial\Delta(c_0, r)$ onto $\partial\Delta(0, r) = \partial D(0, r)$, so by making the change of variables $\zeta = \sigma_0(z)$ we obtain that

$$\int_{\partial D} (f \circ \sigma \circ \sigma_0)(z)\, dz = \int_{|\zeta|=r} (f \circ \sigma)(\zeta) \frac{1 - |c_0|^2}{(1 + \overline{c}_0\zeta)^2}\, d\zeta.$$

Therefore (1.2) can be written in the following equivalent form:

$$\int_{|\zeta|=r} (f \circ \sigma)(\zeta) \frac{d\zeta}{(1 + \overline{c}_0\zeta)^2} = 0, \qquad \text{for every } \sigma \in \mathcal{M}. \tag{3.2}$$

Now observe that (3.2) can be rewritten as the following convolution equation in the Möbius group $\mathcal{M}$ (see [BZ2, pp. 598-9] where a similar case is treated):

$$\tilde{f} * \check{\tilde{T}} = 0, \tag{3.3}$$

$T = T_{\partial D}$ being the compactly supported Radon measure on $\mathbb{D}$ defined by

$$T\varphi = \int_{|\zeta|=r} \frac{\varphi(\zeta)}{(1 + \overline{c}_0\zeta)^2} \frac{d\zeta}{2\pi i r} \qquad (\varphi \in C(\mathbb{D})).$$

Observe that $T\varphi = 0$ for every holomorphic function φ on $\mathbb{D}$, and, in particular, T vanishes on the constant functions.

3.3 Recall that the (hyperbolic) derivatives $\frac{\partial^{j+k}\delta_0}{\partial z^j \partial \overline{z}^k}$ of the Dirac delta measure at the origin δ_0 are given by

$$\langle \frac{\partial^{j+k}\delta_0}{\partial z^j \partial \overline{z}^k}, \varphi \rangle = (-1)^{j+k} \frac{\partial^{j+k}}{\partial z^j \partial \overline{z}^k} \left(\frac{\varphi(z)}{(1 - |z|^2)^2} \right)(0) \qquad (\varphi \in \mathcal{E}(\mathbb{D})). \tag{3.4}$$

The corresponding Euclidean derivatives $D_{j,k}\delta_0$ are defined to be

$$\langle D_{j,k}\delta_0, \varphi \rangle = (-1)^{j+k} \frac{\partial^{j+k}\varphi}{\partial z^j \partial \overline{z}^k}(0) \qquad (\varphi \in \mathcal{E}(\mathbb{D})).$$

In order to study whether (3.3) implies that f is harmonic, consider the closed convolution ideal $\mathcal{I}$ in $\mathcal{E}'_0(\mathcal{M})$ generated by all the distributions of the form $\check{\tilde{T}} * (D_{j,k}\delta_0)\tilde{}$, $j, k \geq 0$, and observe that

$$\tilde{f} * R = 0, \qquad \text{for every } R \in \mathcal{I}. \tag{3.5}$$

Let us remark that in the definition of $\mathcal{I}$ we can replace the Euclidean derivatives $D_{j,k}\delta_0$ by the hyperbolic derivatives $\frac{\partial^{j+k}\delta_0}{\partial z^j \partial \bar{z}^k}$. In fact, it is easy to prove the following formula:

$$\frac{\partial^{j+k}\delta_0}{\partial z^j \partial \bar{z}^k} = \sum_{\ell=0}^{\min\{j,k\}} (\ell+1)\frac{j!k!}{(j-\ell)!(k-\ell)!} D_{j-\ell,k-\ell}\delta_0, \tag{3.6}$$

which shows that the two linear subspaces of $\mathcal{E}'(\mathbb{D})$ spanned by the families $\{\frac{\partial^{j+k}\delta_0}{\partial z^j \partial \bar{z}^k}\}_{j,k\geq 0}$ and $\{D_{j,k}\delta_0\}_{j,k\geq 0}$, respectively, coincide.

3.4 Now we recall the concept of spherical function and spherical Fourier transform in $\mathcal{M}/\mathcal{K} \cong \mathbb{D}$.

A **spherical function** on $\mathcal{M}/\mathcal{K}$ is any eigenfunction $\varphi \in C_0^\infty(\mathcal{M})$ of $\tilde{\Delta}_h$ such that $\varphi(e) = 1$. That is, the spherical functions on $\mathcal{M}/\mathcal{K}$ are the liftings to $\mathcal{M}$ of the radial eigenfunctions φ of the hyperbolic Laplacian Δ_h normalized by $\varphi(0) = 1$. Equivalently, $\varphi \in C_0(\mathcal{M})$ is a spherical function on $\mathcal{M}/\mathcal{K}$ if it satisfies the functional equation:

$$\int_{\mathcal{K}} \varphi(gkh)\,dk = \varphi(g)\varphi(h) \qquad (g,h \in \mathcal{M})$$

(see [Hel2, pp. 399–400]).

For every $\lambda \in \mathbb{C}$ there is only one spherical function φ_λ such that $\tilde{\Delta}_h\varphi_\lambda = -(1+\lambda^2)\varphi_\lambda$, which is given by (see either [BZ2, (6.2)] or [Hel2, p. 40])

$$(\varphi_\lambda)_\pi(w) = F\left(\frac{1+i\lambda}{2}, \frac{1-i\lambda}{2}; 1; -\frac{|w|^2}{1-|w|^2}\right) \qquad (w \in \mathbb{D}),$$

where $F(a,b;c;z)$ represents, as usual, the classical hypergeometric function.

The **spherical Fourier transform of** $\psi \in \mathcal{D}_0(\mathcal{M})$ is defined by

$$(\mathcal{F}\psi)(\lambda) = \int_{\mathcal{M}} \psi(g)\varphi_\lambda(g^{-1})\,dg = (\psi * \varphi_\lambda)(e) \qquad (\lambda \in \mathbb{C}),$$

and the **spherical Fourier transform of** $S \in \mathcal{E}_0'(\mathcal{M})$ is defined to be

$$(\mathcal{F}S)(\lambda) = (S * \varphi_\lambda)(e) \qquad (\lambda \in \mathbb{C}).$$

By the theorem of Paley–Wiener–Schwartz, $\mathcal{F}$ is an algebra isomorphism between the convolution algebra $\mathcal{E}_0'(\mathcal{M})$ and the multiplication algebra $\mathbb{E}'$ of all even entire functions (of one complex variable) of exponential type which have polynomial growth on $\mathbb{R}$. The topology of $\mathbb{E}'$ is defined in such a way that $\mathcal{F}$ is a topological isomorphism (see [BZ2, pp. 606-608]).

As we will show later, the proof that the harmonicity of f is implied by (3.3) strongly depends on the fact that $\pm i$ are the only common zeros of the functions in the closed ideal $I = \mathcal{F}(\mathcal{I})$ of $\mathbb{E}'$.

It is easy to show that $\pm i$ are common zeros of the functions in I. In fact, if $S \in \mathcal{E}'(\mathbb{D})$ then

$$\mathcal{F}(\check{\tilde{T}} * \tilde{S})(\lambda) = T\left((\tilde{S} * \varphi_\lambda)_\pi\right) \qquad (\lambda \in \mathbb{C}), \tag{3.7}$$

where

$$(\tilde{S} * \varphi_\lambda)_\pi(z) = S\left[\zeta \mapsto (\varphi_\lambda)_\pi\left(\frac{z-\zeta}{1-\bar{\zeta}z}\right)\right] \tag{3.8}$$

and

$$(\varphi_\lambda)_\pi\left(\frac{z-\zeta}{1-\bar{\zeta}z}\right) = F\left(\frac{1+i\lambda}{2}, \frac{1-i\lambda}{2}; 1; \frac{-|z-\zeta|^2}{(1-|z|^2)(1-|\zeta|^2)}\right). \tag{3.9}$$

Therefore

$$(\varphi_{\pm i})_\pi\left(\frac{z-\zeta}{1-\bar{\zeta}z}\right) = 1 \qquad (z, \zeta \in \mathbb{D}),$$

so $(\tilde{S} * \varphi_{\pm i})_\pi$ is a constant function, and hence $\mathcal{F}(\check{\tilde{T}} * \tilde{S})(\pm i) = 0$, for every $S \in \mathcal{E}'(\mathbb{D})$. Finally, since convergence in $\mathbb{E}'$ implies pointwise convergence, it follows that $\pm i$ are common zeros of all the functions in I.

In order to study whether the functions in I have other common zeros, in the next subsections we will find a simple family of generators of I.

3.5 First we are going to show that the derivatives which appear in the definition of $\mathcal{I}$ can be replaced by powers of the hyperbolic Laplacian composed with derivatives with respect to either z or $\bar{z}$.

Lemma 2: *For every $j, k \geq 0$, the distribution $\frac{\partial^{j+k}\delta_0}{\partial z^j \partial \bar{z}^k}$ is a linear combination of the distributions*

$$\Delta_h^\ell\left(\frac{\partial^n \delta_0}{\partial z^n}\right) \qquad (0 \leq \ell \leq \min\{j,k\},\ 0 \leq n \leq j-\ell) \tag{3.10}$$

and

$$\Delta_h^\ell\left(\frac{\partial^n \delta_0}{\partial \bar{z}^n}\right) \qquad (0 \leq \ell \leq \min\{j,k\},\ 0 \leq n \leq k-\ell). \tag{3.11}$$

Taking into account (3.4) and the identities

$$\begin{aligned} \langle \Delta_h^\ell\left(\tfrac{\partial^n\delta_0}{\partial z^n}\right), \varphi\rangle &= (-1)^n \tfrac{\partial^n}{\partial z^n}\left(\Delta_h^\ell \varphi\right)(0) \\ \langle \Delta_h^\ell\left(\tfrac{\partial^n\delta_0}{\partial \bar{z}^n}\right), \varphi\rangle &= (-1)^n \tfrac{\partial^n}{\partial \bar{z}^n}\left(\Delta_h^\ell \varphi\right)(0) \end{aligned} \qquad (\varphi \in \mathcal{E}(\mathbb{D})), \tag{3.12}$$

it is clear that Lemma 2 reduces to the following computational lemma whose proof (which we omit) is done by induction.

LEMMA 3: *Let $j, k \geq 1$ be integers. For every $\varphi \in \mathcal{E}(\mathbb{D})$, $\frac{\partial^{j+k}\varphi}{\partial z^j \partial \overline{z}^k}$ is an $\mathcal{E}(\mathbb{D})$-linear combination of the functions*

$$\frac{\partial^n}{\partial z^n}\left(\Delta_h^\ell \varphi\right) \qquad (1 \leq \ell \leq \min\{j,k\},\ 0 \leq n \leq j-\ell)$$

and

$$\frac{\partial^n}{\partial \overline{z}^n}\left(\Delta_h^\ell \varphi\right) \qquad (1 \leq \ell \leq \min\{j,k\},\ 0 \leq n \leq k-\ell),$$

whose coefficients are functions in $\mathcal{E}(\mathbb{D})$ which do not depend on φ (they only depend on j and k).

Conversely, any of the distributions (3.10) or (3.11) are linear combinations of hyperbolic derivatives of δ_0. In fact, taking into account (3.4) and (3.12), we only have to show that, for every $\varphi \in \mathcal{E}(\mathbb{D})$ and $\ell \geq 0$, $\Delta_h^\ell \varphi$ is an $\mathcal{E}(\mathbb{D})$-linear combination of derivatives $\frac{\partial^{j+k}}{\partial z^j \partial \overline{z}^k}\left(\frac{\varphi(z)}{(1-|z|^2)^2}\right)$ (the coefficients being independent of φ, only dependent on ℓ). And, once again, this is easily proved by induction.

From the above discussion it follows that $I = \mathcal{F}(\mathcal{I})$ is the closed ideal of $\mathbb{E}'$ generated by the functions

$$h_{j,\ell} = \mathcal{F}\left(\check{\tilde{T}} * \left[\Delta_h^\ell\left(\frac{\partial^j \delta_0}{\partial z^j}\right)\right]^{\sim}\right) \qquad (j, \ell \geq 0)$$

and

$$f_{j,\ell} = \mathcal{F}\left(\check{\tilde{T}} * \left[\Delta_h^\ell\left(\frac{\partial^j \delta_0}{\partial \overline{z}^j}\right)\right]^{\sim}\right) \qquad (j, \ell \geq 0).$$

3.6 Next we will show that I can be generated using only the derivatives of δ_0 with respect to either z or $\overline{z}$, so we get rid of the powers of the hyperbolic Laplacian.

For every $S \in \mathcal{E}'(\mathbb{D})$, observe that

$$\widetilde{\Delta_h S} * \varphi_\lambda = \widetilde{\Delta}_h \widetilde{S} * \varphi_\lambda = \widetilde{S} * \widetilde{\Delta}_h \varphi_\lambda = -(1+\lambda^2)(\widetilde{S} * \varphi_\lambda),$$

where the second identity follows from [Hel2, Thm. II.5.5], since Δ_h is an $\mathcal{M}$-invariant differential operator on the symmetric space $\mathcal{M}/\mathcal{K} \cong \mathbb{D}$. By convolving on the left side with $\check{\tilde{T}}$ we have that

$$\mathcal{F}(\check{\tilde{T}} * \widetilde{\Delta_h S})(\lambda) = -(1+\lambda^2)\,\mathcal{F}(\check{\tilde{T}} * \widetilde{S})(\lambda) \qquad (\lambda \in \mathbb{C},\ S \in \mathcal{E}'(\mathbb{D})).$$

Applying the above formula ℓ times with $S=\frac{\partial^j\delta_0}{\partial z^j},\ \frac{\partial^j\delta_0}{\partial \bar{z}^j}$ we obtain that

$$h_{j,\ell}(\lambda)=(-1)^\ell(1+\lambda^2)^\ell\,\mathcal{F}\left(\check{\tilde{T}}*\left(\frac{\partial^j\delta_0}{\partial z^j}\right)^{\sim}\right)(\lambda)$$

and

$$f_{j,\ell}(\lambda)=(-1)^\ell(1+\lambda^2)^\ell\,\mathcal{F}\left(\check{\tilde{T}}*\left(\frac{\partial^j\delta_0}{\partial \bar{z}^j}\right)^{\sim}\right)(\lambda).$$

Therefore, it turns out that I is the closed ideal of $\mathbb{E}'$ generated by the functions

$$h_j=\mathcal{F}\left(\check{\tilde{T}}*\left(\frac{\partial^j\delta_0}{\partial z^j}\right)^{\sim}\right)\qquad (j\ge 0)\tag{3.13}$$

and

$$f_j=\mathcal{F}\left(\check{\tilde{T}}*\left(\frac{\partial^j\delta_0}{\partial \bar{z}^j}\right)^{\sim}\right)\qquad (j\ge 0).\tag{3.14}$$

3.7 Next we will compute explicitly the generators h_j and f_j, $j\ge 0$, of I.

First observe that the hyperbolic derivatives $\frac{\partial^j\delta_0}{\partial z^j}$ and $\frac{\partial^j\delta_0}{\partial \bar{z}^j}$ coincide with the Euclidean derivatives $D_{j,0}\delta_0$ and $D_{0,j}\delta_0$, respectively (see (3.6)). Then, letting $S=\frac{\partial^j\delta_0}{\partial z^j},\ \frac{\partial^j\delta_0}{\partial \bar{z}^j}$ in (3.8) and taking into account (3.9), a straightforward computation using the Faà di Bruno formula shows that

$$\left[\left(\frac{\partial^j\delta_0}{\partial z^j}\right)^{\sim}*\varphi_\lambda\right]_\pi(z)=\left(\frac{\bar{z}}{|z|^2-1}\right)^j F^{(j)}\left(\frac{|z|^2}{|z|^2-1}\right),\tag{3.15}$$

and

$$\left[\left(\frac{\partial^j\delta_0}{\partial \bar{z}^j}\right)^{\sim}*\varphi_\lambda\right]_\pi(z)=\left(\frac{z}{|z|^2-1}\right)^j F^{(j)}\left(\frac{|z|^2}{|z|^2-1}\right),\tag{3.16}$$

where $F(z)=F\left((1+i\lambda)/2,(1-i\lambda)/2;1;z\right)$.

Therefore by (3.7) we have that

$$\begin{aligned} h_j &= F^{(j)}\left(\frac{r^2}{r^2-1}\right)\left(\frac{r}{r^2-1}\right)^j\int_0^{2\pi}\frac{e^{-i(j-1)\theta}}{(1+\bar{c}_0re^{i\theta})^2}\frac{d\theta}{2\pi}\\ &= \begin{cases} 0 & \text{if } j=0\\ -j(\bar{c}_0r)^{j-1}\left(\frac{r}{1-r^2}\right)^j F^{(j)}\left(\frac{r^2}{r^2-1}\right) & \text{if } j\ge 1\end{cases}\end{aligned}$$

and

$$f_j=F^{(j)}\left(\frac{r^2}{r^2-1}\right)\left(\frac{r}{r^2-1}\right)^j\int_0^{2\pi}\frac{e^{i(j+1)\theta}}{(1+\bar{c}_0re^{i\theta})^2}\frac{d\theta}{2\pi}=0\qquad (j\ge 0).$$

Hence, considering the functions

$$(3.17)\quad g_j(\lambda) = F^{(j)}\left(\frac{r^2}{r^2-1}\right) = \frac{d^j}{dz^j}F\left(\frac{1+i\lambda}{2},\frac{1-i\lambda}{2};1;z\right)\Bigg|_{z=\frac{r^2}{r^2-1}} \quad (j \geq 0),$$

we have just shown that I is generated by the g_j, $j \geq 1$, if $c_0 \neq 0$, that is, if $c \neq 0$ (see (2.4)), and it is generated by g_1 if $c = 0$.

Finally note that the function g_1 vanishes at $\pm i$ with multiplicity exactly equal to 1, so, in any case, the "common multiplicity" of the common zeros $\pm i$ in I is equal to 1. In fact, by [E, p. 102 (20)],

$$(3.18)\qquad g_1(\lambda) = \frac{1+\lambda^2}{4}F\left(\frac{3+i\lambda}{2},\frac{3-i\lambda}{2};2;\frac{r^2}{r^2-1}\right) = \frac{1+\lambda^2}{4}G(\lambda),$$

and, by [E, p. 101, 2.8(4)], we have that

$$G(\pm i) = F\left(1,2;2;\frac{r^2}{r^2-1}\right) = 1 - r^2 > 0.$$

3.8 In this subsection and the next one we are going to study the case $c \neq 0$. In that case, we claim that I has only two generators: $g_1(\lambda)$ and $(1+\lambda^2)g_0(\lambda)$.

First of all, observe that the function $(1+\lambda^2)g_0(\lambda)$ belongs to the ideal I. In fact, since F satisfies the hypergeometric equation

$$z(1-z)F''(z) + (1-2z)F'(z) - \frac{1+\lambda^2}{4}F(z) = 0 \qquad (\operatorname{Re} z < 0),$$

it follows that

$$(3.19)\qquad (1+\lambda^2)g_0(\lambda) = \frac{4r^2}{(1-r^2)^2}g_2(\lambda) - 4\frac{1+r^2}{1-r^2}g_1(\lambda).$$

We finish the proof of the claim by showing that

$$(3.20)\qquad g_j(\lambda) = p_j(\lambda)g_1(\lambda) + q_j(\lambda)(1+\lambda^2)g_0(\lambda),$$

for every $j \geq 1$, where p_j and q_j are polynomials.

We proceed by induction on j. For $j = 1$, (3.20) is obvious, while for $j = 2$ it is just another way of writing (3.19).

Now assume (3.20) holds for $j = k-1, k-2$ ($k \geq 2$). We need to know that F satisfies the ordinary differential equation

$$(3.21)\quad z(1-z)F^{(k)} + (k-1)(1-2z)F^{(k-1)} - \frac{(2k-3)^2+\lambda^2}{4}F^{(k-2)} = 0$$

on the half-plane $\operatorname{Re} z < 0$. Observe that (3.21) easily follows since, by [E, p. 102 (20)],

$$F^{(k-2)}(z) = \left(\frac{1+i\lambda}{2}\right)_{(k-2)} \left(\frac{1-i\lambda}{2}\right)_{(k-2)} \frac{1}{(k-2)!} u(z),$$

where

$$u(z) = F\left(\frac{1+i\lambda}{2} + k - 2, \frac{1-i\lambda}{2} + k - 2; k-1; z\right),$$

which satisfies the hypergeometric equation

$$z(1-z)\, u''(z) + (k-1)(1-2z)\, u'(z) - \frac{(2k-3)^2 + \lambda^2}{4} u(z) = 0.$$

Then, using the induction hypothesis and (3.21), (3.20) follows for $j = k$ with

$$p_k(\lambda) = (k-1)\frac{1-r^4}{r^2} p_{k-1}(\lambda) - \frac{(2k-3)^2 + \lambda^2}{4} \frac{r^2}{(1-r^2)^2} p_{k-2}(\lambda)$$

and

$$q_k(\lambda) = (k-1)\frac{1-r^4}{r^2} q_{k-1}(\lambda) - \frac{(2k-3)^2 + \lambda^2}{4} \frac{r^2}{(1-r^2)^2} q_{k-2}(\lambda).$$

Finally we are going to show that the two generators of I, $g_1(\lambda)$ and $(1 + \lambda^2) g_0(\lambda)$, have no common zeros except $\pm i$, and thus $\pm i$ are the only common zeros of all the functions in I.

In fact, assume that $g_1(\lambda_0) = (1 + \lambda_0^2) g_0(\lambda_0) = 0$, for some $\lambda_0 \neq \pm i$. Then it is clear that $g_1(\lambda_0) = g_0(\lambda_0) = 0$, and, by (3.20), it follows that $g_j(\lambda_0) = 0$, for every $j \geq 0$. Taking into account (3.17), this means that the hypergeometric function $F(z) = F((1+i\lambda)/2, (1-i\lambda)/2; 1; z)$ and all its derivatives vanish at $z = r^2/(r^2 - 1)$. Therefore, by analytic continuation, we obtain that F is identically zero, which is clearly absurd.

3.9 By 3.3 every entire function $g(\lambda)$ in I is divisible by $1 + \lambda^2$, so $G(\lambda) = g(\lambda)/(1+\lambda^2)$ is an entire function. Thus, by the theorem of Lindelöf–Malgrange–Ehrenpreis (see either [Ka, p. 135] or [Ko, p. 22]), $G(\lambda)$ is an entire function of exponential type. Moreover, since $g(\lambda)$ is an even function which has polynomial growth on $\mathbb{R}$, it is clear that $G(\lambda)$ also has these two properties. Therefore

$$I_0 = \{\, G(\lambda) = \frac{g(\lambda)}{1+\lambda^2} : g \in I \,\}$$

is a closed ideal in $\mathbb{E}'$.

Furthermore, since the "common multiplicity" of the common zeros $\pm i$ in I is equal to 1 (see 3.7), the functions in I_0 have no common zeros. Therefore, by the spectral synthesis theorem of Schwartz (see [BZ2, p. 608]), the ideal I_0 is dense in $\mathbb{E}'$, so, since it is closed, $I_0 = \mathbb{E}'$.

Finally, we will show that the identity $I_0 = \mathbb{E}'$ implies the harmonicity of any function $f \in C(\mathbb{D})$ satisfying (3.5).

In fact, $I = (1+\lambda^2) \cdot I_0 = (1+\lambda^2) \cdot \mathbb{E}'$ so $1+\lambda^2 \in I$, and therefore $\widetilde{\Delta}_h \delta_{\mathcal{K}} = -\mathcal{F}^{-1}(1+\lambda^2) \in \mathcal{I}$. Then (3.5) implies that $\tilde{f} * \widetilde{\Delta}_h \delta_{\mathcal{K}} = 0$. But $\tilde{f} * \widetilde{\Delta}_h \delta_{\mathcal{K}} = \widetilde{\Delta}_h \tilde{f} * \delta_{\mathcal{K}}$, by [Hel2, Thm. II.5.5], and $\langle \widetilde{\Delta_h f} * \delta_{\mathcal{K}}, \varphi \rangle = \langle \Delta_h f, \varphi_\pi \rangle$, for every $\varphi \in \mathcal{D}(\mathcal{M})$, since $\widetilde{\Delta}_h \tilde{f} = \widetilde{\Delta_h f}$. Thus $\tilde{f} * \widetilde{\Delta}_h \delta_{\mathcal{K}} = 0$ means that $\Delta_h f = 0$, that is, f is harmonic on $\mathbb{D}$. We have just proved part a) of Theorem 1.

3.10 Now we are going to prove part b) of Theorem 1, so we assume $c = 0$. Then we recall that I is generated by g_1 (see (3.18)) with $r = R$, by (2.4).

The first half of part b) follows from the fact that g_1 has some zero $\lambda_0 \neq \pm i$, and then $\lambda_0 \neq \pm i$ is a common zero of all the functions in I.

Observe that, since g_1 vanishes at $\pm i$ with multiplicity equal to 1, we only have to show that the function G that appears in the factorization (3.18) of g_1 has some zero.

Using the Hadamard factorization theorem and [Hol, Thm. III.3.1] it is easy to check that every nonvanishing entire function h of exponential type with polynomial growth on $\mathbb{R}$ has the form

$$h(z) = e^{i\alpha z + \beta} \qquad (z \in \mathbb{C}),$$

where $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{C}$. But any nonconstant such function h is not even. Thus, since our G is a nonconstant even entire function of exponential type with polynomial growth on $\mathbb{R}$ (by the Lindelöf–Malgrange–Ehrenpreis theorem), G must have some zero.

Therefore we have just shown that g_1 has some zero $\lambda_0 \neq \pm i$, so

$$\mathcal{F}R(\lambda_0) = 0 \qquad (R \in \mathcal{I}). \tag{3.22}$$

We finish the proof of the first half of Theorem 1.b), by showing that (3.22) is equivalent to

$$\varphi_{\lambda_0} * \check{\tilde{T}} = 0, \tag{3.23}$$

which means (recall (3.3)) that the nonconstant radial real analytic function $f = (\varphi_{\lambda_0})_\pi$ on $\mathbb{D}$ satisfies (1.2).

In fact, since φ_{λ_0} is a real analytic function on $\mathcal{M}$, so are $\varphi_{\lambda_0} * \check{\tilde{T}}$ and $(\varphi_{\lambda_0} * \check{\tilde{T}})\check{}$, and therefore $\psi = [(\varphi_{\lambda_0} * \check{\tilde{T}})\check{}\,]_\pi$ is a real analytic function on $\mathbb{D}$. Hence, by analytic continuation, (3.23) is equivalent to the vanishing of all the derivatives of ψ at the origin, which is just the meaning of (3.22) because

$$\begin{aligned}\mathcal{F}\left(\check{\tilde{T}} * (D_{j,k}\delta_0)\check{}\right)(\lambda_0) &= \left[\left(\varphi_{\lambda_0} * \check{\tilde{T}}\right) * (D_{j,k}\delta_0)\check{}\,\right](e) \\ &= \left\langle (D_{j,k}\delta_0)\check{}\ ,\ \left(\varphi_{\lambda_0} * \check{\tilde{T}}\right)\check{}\ \right\rangle \\ &= (-1)^{j+k}\frac{\partial^{j+k}\psi}{\partial z^j \partial \bar{z}^k}(0).\end{aligned}$$

3.11 The remaining part of Theorem 1.b) is proved by following the ideas in the proof of part a).

For every $j \in J$, let $\mathcal{I}_j$ be the ideal associated to the distribution $T_j = T_{\partial D(0,R_j)}$ as defined in 3.3. Consider now the closed ideal $\mathcal{I}$ in $\mathcal{E}_0'(\mathcal{M})$ generated by $\cup_{j\in J}\mathcal{I}_j$. Then the fact that $f \in C(\mathbb{D})$ satisfies (1.2), for every $j \in J$, implies that (3.5) holds. Recall that the "common multiplicity" of the common zeroes $\pm i$ of the functions in $I = \mathcal{F}(\mathcal{I})$ is equal to 1, since this is true for every $I_j = \mathcal{F}(\mathcal{I}_j)$ (see 3.7). Therefore, as we proved in 3.9, if the functions in I have no common zeros except $\pm i$, the function f is harmonic. Conversely (see 3.10), if $\lambda \in \mathbb{C}$, $\lambda \neq \pm i$, is a common zero of the functions in I then $(\varphi_\lambda)_\pi$ is a nonconstant radial real analytic function on $\mathbb{D}$ which satisfies (1.2).

Hence to finish the proof of the theorem we only have to show that the absence of common zeros of the functions in I, different from $\pm i$, is equivalent to the absence of common solutions of the equations (1.3).

In fact, recall that we have proved that the closed ideal I_j is generated by the function

$$g_{1,j}(\lambda) = \frac{1+\lambda^2}{4}F_j(\lambda) \qquad (\lambda \in \mathbb{C}),$$

where

$$F_j(\lambda) = F\left(\frac{3+i\lambda}{2}, \frac{3-i\lambda}{2}; 2; \frac{R_j^2}{R_j^2-1}\right).$$

So I is generated by the functions $g_{1,j}$, $j \in J$, and, since $F_j(\pm i) \neq 0$, the common zeros different from $\pm i$ of the functions in I are just the common zeros of the functions F_j, $j \in J$.

But, by [E, p. 140, 3.2(7); p. 148, 3.6(1)], we have that

$$P_z^{-1}(x) = \frac{1}{2}(x^2-1)^{\frac{1}{2}} F\left(\frac{3+i\lambda}{2}, \frac{3-i\lambda}{2}; 2; \frac{1-x}{2}\right) \qquad (x > 1),$$

where $z = \frac{-1+i\lambda}{2}$, $\lambda \in \mathbb{C}$. Then it follows that $F_j(\lambda) = 0$ for every $j \in J$ if and only if z is a common solution of (1.3). The proof of Theorem 1 is complete.

4. Proof of the general Morera theorem

Our next goal is to prove Theorem 2. We will try to imitate the approach followed in the circular case. So the steps which also work in the general case with the same proof will be only sketchily mentioned, and we will concentrate our attention on the new steps or those that require new proofs.

4.1 We would like to point out that, as in the circular case, the method of proof we are going to use can only determine whether a function is harmonic, or more generally, satisfies $\mathcal{M}$-invariant differential equations. For that reason, the following lemma is essential in the proof of Theorem 2.

LEMMA 4: *If f is an harmonic function on $\mathbb{D}$ that satisfies* (1.4) *then f is holomorphic on $\mathbb{D}$.*

Proof: If f is harmonic on $\mathbb{D}$ then $\operatorname{Re} f$, $\operatorname{Im} f$ are real-valued harmonic functions on $\mathbb{D}$, and therefore they are the real parts of some holomorphic functions on $\mathbb{D}$, say

$$g(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad h(z) = \sum_{n=0}^{\infty} b_n z^n,$$

respectively. So

$$f(z) = \frac{g(z) + \overline{g(z)}}{2} + i\,\frac{h(z) + \overline{h(z)}}{2} = \sum_{n=0}^{\infty} c_n z^n + \sum_{n=1}^{\infty} d_n \bar{z}^n = f_1(z) + f_2(z),$$

where

$$c_0 = \operatorname{Re} a_0 + i \operatorname{Re} b_0, \quad c_n = \frac{a_n + ib_n}{2}, \quad d_n = \frac{\bar{a}_n + i\bar{b}_n}{2} \quad (n \geq 1),$$

and these series are uniformly convergent on the compact subsets of $\mathbb{D}$. In particular, f_1 is holomorphic on $\mathbb{D}$ and f_2 is anti-holomorphic on $\mathbb{D}$ and $f_2(0) = 0$. Therefore f is holomorphic on $\mathbb{D}$ if and only if $f_2 \equiv 0$.

By hypothesis, we know that, for every $\sigma \in \mathcal{M}$,

$$\begin{aligned} 0 &= \int_\Gamma (f \circ \sigma)\, dz \\ &= \int_\Gamma (f_1 \circ \sigma)\, dz + \int_\Gamma (f_2 \circ \sigma)\, dz \end{aligned}$$

$$= \int_\Gamma (f_2 \circ \sigma)\, dz$$
$$= \sum_{n=1}^{\infty} d_n \int_\Gamma \left(\overline{\sigma(z)}\right)^n dz,$$

and this series converges absolutely. (Note that the third of the above identities follows from Cauchy's theorem and the holomorphy of $f_1 \circ \sigma$.)

For the rotation $\sigma(z) = e^{-i\theta} z$ ($\theta \in \mathbb{R}$), we have

$$0 = \sum_{n=1}^{\infty} d_n \left(\int_\Gamma \overline{z}^n\, dz \right) e^{in\theta},$$

the convergence of the series being absolutely and uniformly in $\theta \in \mathbb{R}$. Therefore

$$0 = d_1 \left(\int_\Gamma \overline{z}\, dz \right) = d_1 \int_\Omega d\overline{z} \wedge dz = 2id_1 \int_\Omega dx\, dy = 2id_1 \operatorname{area}(\Omega),$$

where the second identity follows from Stokes theorem. Since obviously Ω has positive area we obtain that $0 = d_1 = \partial f_2/\partial\overline{z}(0)$. Then the last argument in the proof of Lemma 1 (applied to f_2 instead of f) shows that $\partial f_2/\partial\overline{z} \equiv 0$, and that means f_2 is constant (recall that f_2 is anti-holomorphic.) Finally, since $f_2(0) = 0$, we conclude that $f_2 \equiv 0$, i.e., f is holomorphic.

4.2 Next we rewrite (1.4) as the convolution equation (3.3), where now $T = T_\Gamma$ is the compactly supported Radon measure defined by

$$T\varphi = \int_\Gamma \varphi(\zeta)\, d\zeta \qquad (\varphi \in C(\mathbb{D})). \tag{4.1}$$

Then the closed convolution ideal $\mathcal{I}$ in $\mathcal{E}_0'(\mathcal{M})$ generated by all the bi-invariant distributions $\check{\tilde{T}} * \tilde{S}$, where S runs through $\mathcal{E}'(\mathbb{D})$, satisfies (3.5).

4.3 We cannot continue following the "circular" approach due to the impossibility of computing explicitly the spherical Fourier transforms of (even simple) distributions in $\mathcal{I}$, and then studying their common zeros. So in order to complete the proof of Theorem 2 we will follow a more indirect path which is inspired by the one followed by Berenstein and Shahshahani in [BS] while working on the Pompeiu problem.

Since T obviously vanishes on the holomorphic functions, the computations made in the circular case (see 3.4) show that every function in the closed multiplication ideal $I = \mathcal{F}(\mathcal{I})$ of $\mathbb{E}'$ vanishes at $\pm i$. By 3.9 we know that if the only common zeros of the functions in that ideal I are $\pm i$, and their multiplicities for at least one of such functions are equal to 1, then f is an harmonic function on $\mathbb{D}$. Thus, if we assume that f is not harmonic, we obtain that either there is some common zero $\lambda_0 \in \mathbb{C} \setminus \{\pm i\}$ of all the functions in I, or every function in I vanishes at $\pm i$ with multiplicity greater than or equal to 2. In the next subsection we will see that the last possibility does not happen, and, moreover, the above common zero $\lambda_0 \neq \pm i$ cannot be arbitrary.

4.4 For every $\lambda \in \mathbb{C}$, let $\mathcal{P}_\lambda$ be the following complex power of the classical Poisson kernel on $\mathbb{D}$:

$$\mathcal{P}_\lambda(z,\zeta) = \left(\frac{1-|z|^2}{|z-\zeta|^2}\right)^{\frac{1+i\lambda}{2}} \qquad (z \in \mathbb{D},\ \zeta \in \partial\mathbb{D}).$$

We recall that the function $\mathcal{P}_\lambda(\cdot,\zeta)$ is an eigenfunction of the hyperbolic Laplacian (see [Hel2, p. 32, Lemma 4.1]), namely:

$$\text{(4.2)} \qquad \Delta_h \mathcal{P}_\lambda(\cdot,\zeta) = -(1+\lambda^2)\mathcal{P}_\lambda(\cdot,\zeta), \qquad \text{for every } \zeta \in \partial\mathbb{D}.$$

We also recall that a point $\lambda \in \mathbb{C}$ is called **simple** if the mapping $g \in L^2(\partial\mathbb{D}) \mapsto g^\star \in C^\infty(\mathbb{D})$ given by

$$g^\star(z) = \int_{\partial\mathbb{D}} \mathcal{P}_\lambda(z,\zeta) g(\zeta)\,|d\zeta|$$

is one-to-one.

It is known that the nonsimple points are the complex points $i(1+2k)$, for $k \in \mathbb{Z}^+$ (see [Hel, p. 46, Prop. 4.8].)

We are going to show that the function $h = \mathcal{F}(\check{\check{T}} * (\widetilde{\frac{\partial \delta_0}{\partial z}})) \in I$ does not vanish at any nonsimple point $\lambda \in \mathbb{C}$, $\lambda \neq i$, and its zeros $\pm i$ have multiplicity equal to 1. Therefore there exists a common zero $\lambda_0 \in \mathbb{C} \setminus \{\pm i\}$ of all the functions in I, which necessarily is a simple point.

First observe that, by (4.1) and Stokes' theorem,

$$\text{(4.3)} \qquad T\varphi = 2i \int_\Omega \frac{\partial \varphi}{\partial \bar z}(x+iy)\,dx\,dy \qquad (\varphi \in \mathcal{E}(\mathbb{D})).$$

Now note that (3.7), (3.8) and (3.9) hold, so by differentiating (3.16) we obtain

$$\frac{\partial}{\partial \bar z}\left(\left(\widetilde{\frac{\partial \delta_0}{\partial z}}\right) * \varphi_\lambda\right)_\pi (z) = \frac{-1}{(1-|z|^2)^2}\,[F'(t) + t\,F''(t)]_{t=\frac{|z|^2}{|z|^2-1}}$$

Thus, by (3.7) and (4.3),

$$\begin{aligned} h(\lambda) &= -2i\int_0^{2\pi}\left\{\int_{\Omega_\theta}\left[F'\left(\frac{r^2}{r^2-1}\right)-\frac{r^2}{1-r^2}F''\left(\frac{r^2}{r^2-1}\right)\right]\frac{r\,dr}{(1-r^2)^2}\right\}d\theta \\ &= -i\int_0^{2\pi}\left\{\int_{\Omega'_\theta}\left[F'\left(\frac{r}{r-1}\right)+\frac{r}{r-1}F''\left(\frac{r}{r-1}\right)\right]\frac{dr}{(1-r)^2}\right\}d\theta, \end{aligned}$$

where

$$\begin{aligned} \Omega_\theta &= \{\,0<r<1\colon re^{i\theta}\in\Omega\,\} \\ \Omega'_\theta &= \{\,0<r<1\colon \sqrt{r}e^{i\theta}\in\Omega\,\} \end{aligned} \qquad (0\le\theta\le 2\pi).$$

By making the change of variable $x = r/(r-1)$ we have

$$h(\lambda) = -i\int_0^{2\pi}\left\{\int_{\Omega''_\theta}(F'(x)+xF''(x))\,dx\right\}d\theta,$$

where

$$\Omega''_\theta = \{\,x<0\colon \sqrt{\frac{x}{x-1}}e^{i\theta}\in\Omega\,\} \qquad (0\le\theta\le 2\pi).$$

Now, by [E, p. 58 (7)],

$$F'(x)+xF''(x) = \frac{d}{dx}(xF'(x)) = \frac{1+\lambda^2}{4}\frac{d}{dx}\left(xF\left(\frac{3+i\lambda}{2},\frac{3-i\lambda}{2};2;x\right)\right).$$

So $h(\lambda) = -i\frac{1+\lambda^2}{4}h_0(\lambda)$, where

$$h_0(\lambda) = \int_0^{2\pi}\left\{\int_{\Omega''_\theta}\frac{d}{dx}\left(xF\left(\frac{3+i\lambda}{2},\frac{3-i\lambda}{2};2;x\right)\right)dx\right\}d\theta \qquad (\lambda\in\mathbb{C}).$$

Thus it is clear we only have to show that $h_0(i(1+2k))\neq 0$, for every $k\in\mathbb{Z}^+$. In fact, for $k\in\mathbb{Z}^+$ we have

$$h_0(i(1+2k)) = \int_0^{2\pi}\left\{\int_{\Omega''_\theta}\frac{d}{dx}(xF(1-k,2+k;2;x))\,dx\right\}d\theta, \tag{4.4}$$

and we are going to prove the above assertion by showing that the derivative in (4.4) is positive.

For $k=0$ that derivative is

$$\frac{d}{dx}(xF(1,2;2;x)) = \frac{d}{dx}\left(\frac{x}{1-x}\right) = \frac{1}{(1-x)^2} > 0 \qquad (x<0).$$

(The first identity follows by analytic continuation since the corresponding hypergeometric series coincides with the geometric series.)

For $k \geq 1$ the hypergeometric series of $F(1-k, 2+k; 2; z)$ terminates and again by analytic continuation it turns out that, for $x < 0$,

$$\begin{aligned} \frac{d}{dx}\left(xF(1-k,2+k;2;x)\right) &= \sum_{n=0}^{k-1} \frac{(1-k)_n(2+k)_n}{(2)_n n!}(n+1)x^n \\ &= \sum_{n=0}^{k-1} \binom{k-1}{n}\binom{k+n+1}{n}(-x)^n \\ &> 0, \end{aligned}$$

which completes the proof of the assertion.

4.5 The **Helgason Fourier transform** of a compactly supported function $\varphi \in L^1(\mathbb{D})$ is given by

$$\mathcal{F}\varphi(\lambda,\zeta) = \int_{\mathbb{D}} \mathcal{P}_{-\lambda}(z,\zeta)\varphi(z)\,d\mu(z) \qquad (\lambda \in \mathbb{C},\ \zeta \in \partial\mathbb{D}).$$

The **Helgason Fourier transform** of $R \in \mathcal{E}'(\mathbb{D})$ is defined by

$$\mathcal{F}R(\lambda,\zeta) = \langle R, \mathcal{P}_{-\lambda}(\cdot,\zeta)\rangle \qquad (\lambda \in \mathbb{C},\ \zeta \in \partial\mathbb{D}).$$

We are going to prove that

$$\mathcal{F}T(\pm\lambda_0,\zeta) = 0, \qquad \text{for every } \zeta \in \partial\mathbb{D}. \tag{4.5}$$

In 3.10 we have shown that λ_0 being a common zero of all the functions in I is equivalent to (3.23). But

$$\left(\varphi_\lambda * \check{\tilde{T}}\right)^{\check{}}(g) = \int_\Gamma (\varphi_\lambda)_\pi(g^{-1}(z))\,dz, \qquad (g \in \mathcal{M},\ \lambda \in \mathbb{C}),$$

and taking into account the formula

$$(\varphi_\lambda)_\pi(g^{-1}(z)) = \int_{\partial\mathbb{D}} \mathcal{P}_{-\lambda}(g(0),\zeta)\,\mathcal{P}_\lambda(z,\zeta)\,|d\zeta|$$

(see [Hel2, p. 45 (35)]) we obtain that

$$\left(\varphi_\lambda * \check{\tilde{T}}\right)^{\check{}}(g) = \int_{\partial\mathbb{D}} \mathcal{P}_{-\lambda}(g(0),\zeta)\,\mathcal{F}T(-\lambda,\zeta)\,|d\zeta|.$$

Therefore (3.23) means that

$$\int_{\partial\mathbb{D}} \mathcal{P}_{-\lambda_0}(z,\zeta)\,\mathcal{F}T(-\lambda_0,\zeta)\,|d\zeta| = 0, \qquad \text{for every } z \in \mathbb{D}.$$

Hence, since $\mathcal{F}T(\pm\lambda_0,\cdot) \in C^\infty(\partial\mathbb{D})$ and, since by 4.4, the points $\pm\lambda_0$ are simple, it follows that (4.5) holds.

4.6 Now we are going to show that (4.5) implies there exists a solution $S \in \mathcal{E}'(\mathbb{D})$ to the equation

$$\Delta_h S - \alpha S = T, \tag{4.6}$$

where $\alpha = -1 - \lambda_0^2$.

Recall that there is an isomorphism γ from the space $\mathrm{Diff}(\mathbb{D})$ of $\mathcal{M}$-invariant differential operators on $\mathbb{D}$ onto the space of even polynomials in one complex variable such that

$$L\left(\mathcal{P}_\lambda(\cdot,\zeta)\right) = \gamma(L)(i\lambda)\,\mathcal{P}_\lambda(\cdot,\zeta) \qquad (z \in \mathbb{D},\ \zeta \in \partial\mathbb{D}),$$

for every $L \in \mathrm{Diff}(\mathbb{D})$ (see [Hel4, pp. 10,94].)

Let $L_0 = \Delta_h + 1 + \lambda_0^2$. Then by (4.2) it is clear that $\gamma(L_0)(i\lambda) = -\lambda^2 + \lambda_0^2$, which only vanishes at $\lambda = \pm\lambda_0$. From (4.5) it follows that $\mathcal{F}T(\lambda,\zeta)/\gamma(L_0)(i\lambda)$ is an entire function, for every $\zeta \in \partial\mathbb{D}$. Thus we can apply the Helgason division theorem [Hel3, Thm. 8.5] to obtain the existence of a distribution $S \in \mathcal{E}'(\mathbb{D})$ such that $L_0 S = T$, that is, S satisfies (4.6), with $\alpha = -1 - \lambda_0^2$. Note that $\alpha \neq 0$, since $\lambda_0 \neq \pm i$.

4.7 The next step consists of showing that any solution $S \in \mathcal{E}'(\mathbb{D})$ to the equation (4.6) must be a compactly supported continuous function u on $\mathbb{D}$, whose modulus of continuity

$$\omega(z,\delta) = \sup\{\,|u(z') - u(z)|\colon z' \in \mathbb{D},\ |z' - z| \leq \delta\,\} \qquad (z \in \mathbb{D},\ 0 < \delta)$$

satisfies the following uniform growth condition on Γ:

$$\sup_{z\in\Gamma} \omega(z,\delta) = O\left(\delta \log \frac{1}{\delta}\right), \qquad \text{as } \delta \searrow 0. \tag{4.7}$$

(In the course of the proof we will show that a solution $S \in \mathcal{E}'(\mathbb{D})$ to (4.6) is in fact unique.)

In order to do that, we need to write S in an adequate way.

First of all, we find a fundamental solution N_α on $\mathbb{D}$ of $\Delta_h - \alpha$, that is, N_α is a locally integrable function on $\mathbb{D}$ such that

$$\Delta_h N_\alpha - \alpha N_\alpha = \delta_0. \tag{4.8}$$

As in the classical Euclidean case, the natural candidates for N_α are the radial solutions u to the equation $\Delta_h u = \alpha u$. Taking into account the expression of the hyperbolic Laplacian in geodesic polar coordinates $z = (\tanh r)e^{i\theta}$:

$$\Delta_h = \frac{\partial^2}{\partial r^2} + 2\coth(2r)\frac{\partial}{\partial r} + 4\frac{1}{\sinh^2(2r)}\frac{\partial^2}{\partial \theta^2}$$

(see [Hel2, p. 38(17)]), it is easy to obtain that the general form of those solutions is:

$$u(z) = A \cdot P_\nu(\cosh(2r)) + B \cdot Q_\nu(\cosh(2r)) \qquad (A, B \in \mathbb{C}), \tag{4.9}$$

where $\alpha = \nu(\nu+1)$ and P_ν, Q_ν are the Legendre functions of degree ν of the first and second kinds, respectively. (See [T, p. 269] where this proccess is carried out for the hyperbolic upper half-plane.)

Since $\alpha = -1 - \lambda_0^2$, ν must be $(-1 \pm \lambda_0)/2$. For convenience, and without loss of generality, we may assume $\operatorname{Re}\lambda_0 \geq 0$ and then take $\nu = (\lambda_0 - 1)/2$, so that $\operatorname{Re}\nu \geq -\frac{1}{2}$.

But $P_\nu(\cosh(2r))$ is a C^∞ function on $\mathbb{D}$ (see [L, p. 167]), so the only candidates for N_α are obtained by setting $A = 0$ in (4.9). Then u is a locally integrable function on $\mathbb{D}$ because its singularity at the origin is logarithmic. In fact, we have the estimate

$$Q_\nu(\cosh x) \sim -\log(1 - e^{-x}) \qquad (x \searrow 0), \tag{4.10}$$

which holds for $\operatorname{Re}\nu > -1$ (see [E, 3.9 (7)]) and so for our selection of ν.

Now taking into account that

$$Q'_\nu(x) \sim \frac{1}{1 - x^2} \qquad (x \searrow 1) \tag{4.11}$$

(see [E, 3.6(5), 3.9(9)]), a standard argument based on the second Green's formula shows that

$$N_\alpha(z) = -\frac{1}{2\pi}Q_\nu(\cosh(2r)) = -\frac{1}{2\pi}Q_\nu\left(\frac{1 + |z|^2}{1 - |z|^2}\right) \tag{4.12}$$

is the fundamental solution we are searching for.

Lifting (4.8) from $\mathbb{D}$ to $\mathcal{M}$ we obtain:

$$\widetilde{\Delta}_h \widetilde{N}_\alpha - \alpha \widetilde{N}_\alpha = \delta_{\mathcal{K}}.$$

Then, since $L = \Delta_h - \alpha$ is an $\mathcal{M}$-invariant differential operator on the symmetric space $\mathcal{M}/\mathcal{K} \cong \mathbb{D}$, by [Hel2, Thm. II.5.5] we have that

$$\widetilde{T} * \widetilde{N}_\alpha = (\widetilde{L}\widetilde{S}) * \widetilde{N}_\alpha = \widetilde{S} * (\widetilde{L}\widetilde{N}_\alpha) = \widetilde{S} * \delta_K = \widetilde{S},$$

where $\widetilde{L} = \widetilde{\Delta}_h - \alpha$. Therefore

$$S = (\widetilde{T} * \widetilde{N}_\alpha)_\pi$$

is the only solution in $\mathcal{E}'(\mathbb{D})$ to the equation (4.6). From this expression it is easy to see that S coincides with the locally integrable function

$$u(z) = \int_\Gamma G_\alpha(z,w)\,dw \qquad \text{for a.e. } z \in \mathbb{D}, \tag{4.13}$$

where

$$G_\alpha(z,w) = N_\alpha\left(\frac{z-w}{1-\overline{w}z}\right) \qquad (z,w \in \mathbb{D},\ z \neq w) \tag{4.14}$$

is the Green's function of $\Delta_h - \alpha$.

Finally, we are going to show that the integral in (4.13) defines a continuous function on $\mathbb{D}$ (which we continue calling u). In fact, observe that the above integral defines a C^∞ function on $\mathbb{D} \setminus \Gamma$. Moreover the integral also makes sense for $z \in \Gamma$ because the singularity of $G_\alpha(z,\cdot)$ at $w = z$ is logarithmic. Namely, we have that

$$G_\alpha(z,w) \sim \frac{1}{2\pi}\log \rho(z,w), \qquad \text{as } \rho(z,w) \searrow 0, \tag{4.15}$$

where ρ is the pseudohyperbolic distance (see (2.3)). (The above estimate directly follows from (4.14), (4.12) and (4.10).)

Thus the integral in (4.13) defines u everywhere on $\mathbb{D}$ and u is C^∞ on $\mathbb{D} \setminus \Gamma$, so we only have to show that it satisfies (4.7).

Since $\Gamma \subset\subset \mathbb{D}$, in order to prove (4.7) we can replace in the definition of $\omega(z,\delta)$ the Euclidean distance $|z'-z|$ by the pseudohyperbolic distance $\rho(z',z)$. In fact, if K is a compact subset of $\mathbb{D}$, we have that

$$\rho(z',z) \simeq |z'-z| \qquad (z',\, z \in K). \tag{4.16}$$

Let $z_0 \in \Gamma$ and $z_1 \in \mathbb{D}$ such that $0 < \rho(z_1,z_0) \le \delta < \frac{1}{2}$. Then

$$\begin{aligned} |u(z_1) - u(z_0)| &\le \left\{\int_{\Gamma\cap\Delta_{2\delta}(z_0)} + \int_{\Gamma\setminus\Delta_{2\delta}(z_0)}\right\} |G_\alpha(z_1,w) - G_\alpha(z_0,w)|\,|dw| \\ &= J + J'. \end{aligned}$$

We are going to show that both J and J' are $O(\delta\log(1/\delta))$, as $\delta \searrow 0$.

We estimate J in the following way:

$$J \le \int_{\Gamma\cap\Delta_{2\delta}(z_0)} |G_\alpha(z_0,w)|\,|dw| + \int_{\Gamma\cap\Delta_{2\delta}(z_0)} |G_\alpha(z_1,w)|\,|dw| = J_1 + J_2.$$

Using equation (4.15) we conclude that

$$J_1 \preceq \int_{\Gamma\cap\Delta_{2\delta}(z_0)} \log\frac{1}{\rho(z_0,w)}\,|dw| \qquad (0<\delta<\frac{1}{4}).$$

Obtaining a similar estimate for J_2 is a little more complicate. Note that $\Delta_{2\delta}(z_0)\subset\Delta_{3\delta}(z_1)$. Let $w_1\in\Gamma$ be such that $\rho(z_1,w_1)=\rho(z_1,\Gamma)$. Then we have $\Gamma\cap\Delta_{3\delta}(z_1)\subset\Gamma\cap\Delta_{4\delta}(w_1)$ and $\rho(w,w_1)\le 2\,\rho(w,z_1)$, for every $w\in\Gamma\cap\Delta_{3\delta}(z_1)$. Therefore

$$J_2 \preceq \int_{\Gamma\cap\Delta_{4\delta}(w_1)} \log\frac{1}{\rho(w,w_1)}|dw|, \qquad (0<\delta<\frac{1}{6}).$$

Thus in order to complete the estimate of J we only have to prove that

$$\int_{\Gamma\cap\Delta_{\delta}(z)} \log\frac{1}{\rho(w,z)}\,|dw| \preceq \delta\log\frac{1}{\delta}, \tag{4.17}$$

for $\delta>0$ small and $z\in\Gamma$.

Since Ω is a Lipschitz Jordan domain, the arc length distance between two points w and z in Γ, i.e., the smallest length of the two arcs in Γ joining w and z, is comparable to their Euclidean distance $|w-z|$. The above fact and (4.16) imply that there are constants $C>0$ and $c>1$ such that, for $\delta>0$ small enough and for every $z\in\Gamma$, we have

$$\int_{\Gamma\cap\Delta_{\delta}(z)} \log\frac{1}{\rho(w,z)}\,|dw| \le C\int_0^{c\delta}\log\frac{1}{t}\,dt \le c\,C\,\delta\,\log\frac{1}{\delta}.$$

Now we want to obtain the estimate for J'.

Taking into account (4.11) it is easy to check that, given $0<\rho_0<1$, we have that

$$\left|\frac{\partial G_\alpha}{\partial z}(z,w)\right| \simeq \frac{1}{\rho(z,w)} \simeq \left|\frac{\partial G_\alpha}{\partial \bar z}(z,w)\right|,$$

for $z,w\in\mathbb{D}$, $\rho(z,w)\le\rho_0$.

Moreover,

$$\rho(z_0,w)\le 2\rho(z,w), \qquad \text{for every } w\in\Gamma\setminus\Delta_{2\delta_0}(z_0) \text{ and } z\in\Delta_\delta(z_0).$$

Therefore, by the mean-value theorem and (4.16) we obtain

$$|G_\alpha(z_1,w)-G_\alpha(z_0,w)| \preceq \frac{\rho(z_1,z_0)}{|z_0-w|} \qquad (w\in\Gamma\setminus\Delta_{2\delta}(z_0)).$$

This estimate and the fact that arc length distance and Euclidean distance are comparable on Γ show that there are constants $C, c > 0$ such that for small $\delta > 0$ we have that

$$J' \leq C\left(\int_{c\delta}^{\frac{\ell}{2}} \frac{1}{s}\,ds\right)\cdot \rho(z_1, z_0) \leq C'\cdot\delta\log\frac{1}{\delta},$$

where ℓ is the length of Γ and $C' > 0$ is a constant that only depends on c, C and ℓ. Hence we have just proved the estimate for J', and we have finished the proof of (4.7).

Finally note that the continuous function u is compactly supported in $\mathbb{D}$ because the distribution S is.

4.8 We have just seen in 4.7 that there is a compactly supported continuous function u on $\mathbb{D}$ satisfying

$$\Delta_h u - \alpha u = T, \qquad \text{in the sense of distributions.} \tag{4.18}$$

Since T is supported on Γ, (4.18) implies that

$$\Delta_h u - \alpha u = 0, \qquad \text{on } \mathbb{D}\setminus\Gamma.$$

Therefore u is real analytic on $\mathbb{D}\setminus\Gamma$, because the differential operator $\Delta_h - \alpha$ is elliptic. But u has compact support in $\mathbb{D}$, Ω is relatively compact in $\mathbb{D}$ and $\mathbb{D}\setminus\overline{\Omega}$ is connected, so, by analytic continuation, we have that $u = 0$ on $\mathbb{D}\setminus\overline{\Omega}$. The continuity of u shows that also $u = 0$ on Γ, i.e., $u = 0$ on $\mathbb{D}\setminus\Omega$.

4.9 The next step consists of showing that the constant α is real and, in fact, $\alpha < 0$.

Observe that T belongs to the Sobolev space $H_{-1} = H_{-1}(\mathbb{C})$ (here we use the notation of [F, Ch. 6]). In fact, (4.3) and the Schwarz inequality show that

$$|\langle T,\varphi\rangle| = 2\left|\int_\Omega \frac{\partial\varphi}{\partial\bar z}(z)\,dm(z)\right| \leq 2\sqrt{\pi}\left\|\frac{\partial\varphi}{\partial\bar z}\right\|_{L^2(\mathbb{C})} \qquad (\varphi\in\mathcal{D}(\mathbb{C})).$$

Thus $T\in H_{-1}\subset H_{-1}^{\mathrm{loc}}(\mathbb{D})$, and, since $\Delta_h - \alpha$ is a second order elliptic operator, it follows from (4.18), by the local regularity theorem [F, (6.30)], that $u\in H_1^{\mathrm{loc}}(\mathbb{D})$, which means that u and all its first order (Euclidean) partial derivatives (in the sense of (Euclidean) distributions) are locally square integrable functions on $\mathbb{D}$.

The fact that α is negative is a consequence of the identity

$$\alpha\int_\Omega |u|^2\,d\mu = -\int_\Omega |\nabla_h u|_h^2\,d\mu, \tag{4.19}$$

which we shall now prove, because the above two integrals are finite and positive.

Here $|\cdot|_h$ denotes the hyperbolic norm for vectors in the tangent bundle $T(\mathbb{D})$ of $\mathbb{D}$, which by (2.1) is given in terms of the Euclidean norm by the formula

$$|X|_h^2 = \frac{|X|^2}{(1-|z|^2)^2} \qquad (X \in T_z(\mathbb{D}),\ z \in \mathbb{D}). \tag{4.20}$$

Therefore the hyperbolic gradient $\nabla_h u$ and the Euclidean gradient ∇u of u are related by

$$\nabla_h u(z) = (1-|z|^2)^2 \nabla u(z). \tag{4.21}$$

Thus

$$\int_\Omega |\nabla_h u|_h^2 \, d\mu = \int_\Omega |\nabla u|^2 \, dm.$$

Now, for $\delta > 0$ small, consider the "approximating" open set

$$\Omega_\delta = \{\, z \in \Omega : \operatorname{dist}(z, \Gamma) > \delta \,\}.$$

(Here "dist" means Euclidean distance.)

Take $\varphi^\delta \in \mathcal{D}(\Omega)$ so that $0 \le \varphi^\delta \le 1$, $\varphi^\delta = 1$ on Ω_δ and $|\nabla \varphi^\delta| \le C/\delta$, where C is a positive constant independent of δ (see [Hor, Thm. 1.4.2].) Since $u \in H_1^{\mathrm{loc}}(\mathbb{D})$ we have that

$$\alpha \int_\Omega |u|^2 \, d\mu = \lim_{\delta \searrow 0} \alpha \int_\Omega \varphi^\delta \, |u|^2 \, d\mu$$

and

$$\int_\Omega |\nabla_h u|_h^2 \, d\mu = \lim_{\delta \searrow 0} \int_\Omega \varphi^\delta \, |\nabla u|^2 \, dm.$$

But, taking into account that $\Delta_h u = \alpha u$ on Ω, by integration by parts we obtain that

$$\begin{aligned} \alpha \int_\Omega \varphi^\delta \, |u|^2 \, d\mu &= \int_\Omega \varphi^\delta \, \overline{u} \, \Delta u \, dm \\ &= -\int_\Omega \langle \nabla u, \nabla(\varphi^\delta u) \rangle \, dm \\ &= -\int_\Omega \varphi^\delta \, |\nabla u|^2 \, dm - \int_\Omega \langle \nabla u, \nabla \varphi^\delta \rangle \, \overline{u} \, dm. \end{aligned}$$

So to complete the proof of (4.19) we only have to show that

$$\lim_{\delta \searrow 0} \int_\Omega \langle \nabla u, \nabla \varphi^\delta \rangle \, \overline{u} \, dm = 0. \tag{4.22}$$

Let $\Omega^\delta = \Omega \setminus \Omega_\delta$. Then the properties of φ^δ and the Schwarz inequality imply that

$$\begin{aligned}\left|\int_\Omega \langle \nabla u, \nabla\varphi^\delta\rangle\, \overline{u}\, dm\right| &\le \int_{\Omega^\delta} |\nabla u|\cdot|\nabla\varphi^\delta|\cdot|u|\, dm \\ &\le \frac{C}{\delta}\left(\sup_{\Omega^\delta}|u|\right)\int_{\Omega^\delta}|\nabla u|\, dm \\ &\le \frac{C}{\delta}\left(\sup_{\Omega^\delta}|u|\right)\sqrt{\operatorname{area}(\Omega^\delta)}\left\{\int_\Omega |\nabla u|^2 dm\right\}^{\frac12}.\end{aligned}$$

Since $u = 0$ on Γ and the modulus of continuity of u satisfies (4.7) it follows that there is a constant $C' > 0$ such that

$$|u(z)| \le C' \operatorname{dist}(z,\Gamma) \log\frac{1}{\operatorname{dist}(z,\Gamma)} \le C'\delta \log\frac{1}{\delta},$$

for every $\delta > 0$ small enough and for every $z \in \Omega^\delta$. Moreover, since Γ is a Lipschitz curve, we have that

$$\operatorname{area}(\Omega^\delta) = O(\delta), \qquad \text{as } \delta \searrow 0.$$

Putting all these observations together, we obtain that

$$\int_\Omega \langle \nabla u, \nabla\varphi^\delta\rangle\, \overline{u}\, dm = O\left(\delta^{\frac12}\log\frac{1}{\delta}\right), \qquad \text{as } \delta \searrow 0,$$

from which (4.22) clearly follows.

4.10 Recall that $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ and satisfies

$$\begin{cases} \Delta_h u - \alpha u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma. \end{cases} \tag{4.23}$$

Since by hypothesis Ω is a Jordan domain of class $C^{2,\varepsilon}$, $0 < \varepsilon < 1$, by the regularity theorem [GT, Thm. 6.19] for elliptic operators, u must be of class $C^{2,\varepsilon}$ on $\overline{\Omega}$. Now we are going to get more boundary conditions on u from the equation (4.18).

In fact, taking into account (4.18), that $u = 0$ on $\mathbb{D} \setminus \Omega$ and $u \in C^2(\overline{\Omega})$, by using the (hyperbolic) second Green's formula, we obtain, for every $\varphi \in \mathcal{D}(\mathbb{D})$, that

$$\begin{aligned}\int_\Gamma \varphi(z)\, dz &= \int_\Omega u\,(\Delta_h\varphi - \alpha\varphi)\, d\mu \\ &= \int_\Omega \varphi\,(\Delta_h u - \alpha u)\, d\mu + \int_\Gamma \left(u\,\frac{\partial\varphi}{\partial n_h} - \varphi\,\frac{\partial u}{\partial n_h}\right) ds_h \\ &= -\int_\Gamma \varphi\,\frac{\partial u}{\partial n_h}\, ds_h,\end{aligned}$$

where s_h and n_h denote the hyperbolic arc length of Γ and the hyperbolic unitary outer normal to Γ, respectively. Therefore

$$\int_\Gamma \varphi \frac{\partial u}{\partial n_h} ds_h = -\int_\Gamma \varphi(z)\, dz \qquad (\varphi \in \mathcal{D}(\mathbb{D})). \tag{4.24}$$

But (4.20) shows that the relation between the Euclidean unitary outer normal n to Γ and the hyperbolic one is $n_h = (1-|z|^2)\cdot n$, which together with (4.21) implies that

$$\frac{\partial u}{\partial n_h} = (1-|z|^2)\frac{\partial u}{\partial n}.$$

And, denoting by s the Euclidean arc length of Γ, by (2.2) we conclude that

$$\int_\Gamma \varphi \frac{\partial u}{\partial n_h} ds_h = \int_\Gamma \varphi \frac{\partial u}{\partial n} ds.$$

Hence (4.24) means that

$$\frac{\partial u}{\partial n}(\Gamma(s)) = -\Gamma'(s). \tag{4.25}$$

We want to deduce from the preceding identity some boundary conditions on the partial derivatives of u.

Consider the coordinates $z = x_1 + i\,x_2$. Put $\Gamma(s) = x_1(s) + i\,x_2(s)$ and $u = v + i\,w$. Then $n(s) = n(\Gamma(s)) = x_2'(s) - i\,x_1'(s)$, and (4.25) can be written as

$$\langle \nabla v, n\rangle = -x_1' \quad \text{and} \quad \langle \nabla w, n\rangle = -x_2'.$$

Since $u = 0$ on Γ, we get that ∇v and ∇w are orthogonal to $\Gamma'(s)$ at $\Gamma(s)$, so they are proportional to n at $\Gamma(s)$. And from the above two identities we deduce that

$$\nabla v = -x_1' \cdot n \quad \text{and} \quad \nabla w = -x_2' \cdot n,$$

i.e.,

$$\begin{aligned} \frac{\partial v}{\partial x_1} &= -x_1' x_2', & \frac{\partial v}{\partial x_2} &= (x_1')^2, \\ \frac{\partial w}{\partial x_1} &= -(x_2')^2, & \frac{\partial w}{\partial x_2} &= x_1' x_2'. \end{aligned}$$

In particular, we obtain that

$$\frac{\partial v}{\partial x_1} + \frac{\partial w}{\partial x_2} = 0 \quad \text{and} \quad \frac{\partial v}{\partial x_2} - \frac{\partial w}{\partial x_1} = 1 \quad \text{on } \Gamma.$$

4.11 We know that the real and imaginary parts, v and w, of u are $C^{2,\varepsilon}$ functions on $\overline{\Omega}$, because so is u. Rewriting (4.23) in terms of v and w and adding the boundary conditions we have just obtained, we get that those two real-valued functions are solutions to the following system of elliptic equations with boundary conditions:

$$\left\{\begin{array}{rcl} \Delta_h v - \alpha v &=& 0 \\ \Delta_h w - \alpha w &=& 0 \\ v = w &=& 0 \\ \frac{\partial v}{\partial x_1} + \frac{\partial w}{\partial x_2} &=& 0 \\ \frac{\partial v}{\partial x_2} - \frac{\partial w}{\partial x_1} &=& 1 \end{array}\right. \begin{array}{l} \left.\begin{array}{l} \\ \\ \end{array}\right\} \text{in } \Omega \\ \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} \text{on } \Gamma \end{array} \tag{4.26}$$

Note that, in order to obtain the first two equations from (4.23), we use the fact (proved in 4.9) that α is a real constant.

From now on we will use a semicolon and subscripts to denote partial derivatives, e.g., $h_{;j}$ is the partial derivative of h with respect to the j^{th} variable.

In order to complete the proof of Theorem 2 we will show that we can apply to (4.26) a regularity theorem for elliptic systems with boundary conditions (see [KS, Thm. VI.3.3]) which will imply that Γ is a real analytic curve.

But this regularity result is stated for flat boundaries. So, first, we are going to consider, for every point $z^0 \in \Gamma$, a local change of variables of class $C^{2,\varepsilon}$ which makes Γ "flat" in a neighborhood of that point, that is, in the new variables Γ is just described as a segment of a straight line in a neighborhood of z^0.

Let $z^0 = x_1^0 + ix_2^0 \in \Gamma$. By changing the sign of the variables and the order of the variables and the functions v and w, if it is necessary, we may assume that $v_{;2}(z^0) \neq 0$. (Here we are using the last boundary condition of (4.26).)

Consider the following change of variables, the so-called **zeroth order hodograph transformation** (see [KS, p. 186]):

$$\left\{\begin{array}{rcl} y_1 &=& x_1 - x_1^0 \\ y_2 &=& v \end{array}\right.$$

In fact, the inverse mapping theorem shows it is a local $C^{2,\varepsilon}$ change of variables at z^0 since $v_{;2}(z^0) \neq 0$. (Observe that the y-coordinates of z^0 are $y_1 = y_2 = 0$.) For the same reason, the implicit mapping theorem assures us that the curve Γ is described, in a small neighborhood $\mathcal{U}$ of z^0, by the equation $v(x_1, x_2) = 0$ in the x-coordinates, so it is described by $y_2 = 0$ in the y-coordinates, i.e., $\Gamma \cap \mathcal{U} = \mathcal{U} \cap \{y_2 = 0\}$. Therefore, the part of Ω lying in a small neighborhood of z^0 is described by one of the equations $y_2 > 0$ or $y_2 < 0$.

Note that $x_2 = \psi(y_1, y_2)$, where ψ is a function of class $C^{2,\varepsilon}$ in a neighborhood of the origin. Then a straightforward and tedious calculation shows that the first two equations of (4.26) written in the new variables are:

$$(4.27)\quad 0 = A(y)\left(2\frac{\psi_{;1}}{\psi_{;2}^2}\psi_{;12} - \frac{1}{\psi_{;2}}\psi_{;11} - \frac{1+\psi_{;1}^2}{\psi_{;2}^3}\psi_{;22}\right) - \alpha\, y_2$$

$$(4.28)\quad 0 = A(y)\left(W_{;11} - 2\frac{\psi_{;1}}{\psi_{;2}}W_{;12} + \frac{1+\psi_{;1}^2}{\psi_{;2}^2}W_{;22}\right) + $$
$$+A(y)\left(2\frac{\psi_{;1}}{\psi_{;2}^2}\psi_{;12} - \frac{1}{\psi_{;2}}\psi_{;11} - \frac{1+\psi_{;1}^2}{\psi_{;2}^3}\psi_{;22}\right)W_{;2} - \alpha\, W,$$

where $A(y) = (1-(y_1+x_1^0)^2-\psi^2)^2$ and $W(y) = w(y_1+x_1^0, \psi(y))$, which also is a function of class $C^{2,\varepsilon}$. The above equations (in ψ and W) hold in the intersection of a small neighborhood $\mathcal{U}$ of z^0 with Ω.

The boundary conditions of (4.26) in the new coordinates are:

$$(4.29)\quad W = 0$$

$$(4.30)\quad W_{;2} = \psi_{;1}$$
$$W_{;1} = \frac{1+\psi_{;1}^2}{\psi_{;2}} - 1,$$

and they hold on $\mathcal{U}\cap\{y_2=0\}$.

Observe that the four equations (4.27)–(4.30) do not change formally when you change the sign of both variables. Since from now on we only will use those equations, it follows that we may assume the part of Ω lying in a small neighborhood $\mathcal{U}$ of z^0 is $\mathcal{U}\cap\{y_2>0\}$, i.e., $\Omega\cap\mathcal{U} = \mathcal{U}\cap\{y_2>0\}$.

Thus we can check the hypothesis of the regularity theorem [KS, Thm. VI.3.3] for that system of elliptic equations with boundary conditions.

Since the system (4.27)&(4.28) is clearly nonlinear, we have to compute first the linearization or variational equations (see [KS, p. 192]) of that system and the boundary conditions (4.29)&(4.30):

$$L_{j1}(y,D)\overline{\psi}(y) + L_{j2}(y,D)\overline{W}(y) = 0 \qquad \text{in } \mathcal{U}\cap\{y_2>0\},\ j=1,2$$
$$B_{j1}(y,D)\overline{\psi}(y) + B_{j2}(y,D)\overline{W}(y) = 0 \qquad \text{on } \mathcal{U}\cap\{y_2=0\},\ j=1,2.$$

Here we are following the notation and definitions of [KS, pp. 190–2].

Due to [KS, Thm. VI.3.5], we only have to check the ellipticity and coercivity (with respect to some weights) of the above system of variational equations at the origin.

It turns out that the order of L_{11}, L_{21} and L_{22} is 2, while $L_{12} \equiv 0$. Thus we pick the obvious choice of weights: $s_1 = s_2 = 0$ and $t_1 = t_2 = 2$. Then the "principal symbols"(with respect to those weights) of the L_{jk}'s are:

$$\begin{array}{ll} L'_{11}(y,\xi) = P(y,\xi), & L'_{12}(y,\xi) = 0, \\ L'_{21}(y,\xi) = P(y,\xi)\,W_{;2}, & L'_{22}(y,\xi) = -P(y,\xi)\psi_{;2}, \end{array}$$

where

$$P(y,\xi) = \frac{1}{\psi_{;2}}(1-(y_1+x_1^0)^2-\psi^2)^2\left(\left(\xi_1 - \frac{\psi_{;1}}{\psi_{;2}}\xi_2\right)^2 + \left(\frac{\xi_2}{\psi_{;2}}\right)^2\right).$$

Thus it is clear that the "principal symbol" matrix $(L'_{jk}(y,\xi))$ has rank equal to 2 at the origin, for every $\xi \in \mathbb{R}^2 \setminus \{0\}$.

Moreover, for every pair of independent vectors $\xi, \eta \in \mathbb{R}^2 \setminus \{0\}$ the polynomial

$$p(z) = \det(L'_{jk}(0,\xi+z\eta)) = -\left(P(0,\xi+z\eta)\right)^2 \psi_{;2}(0),$$

with real coefficients, does not vanish on the real line, so it has exactly $\mu = 2$ roots with positive imaginary part and $\mu = 2$ roots with negative imaginary part.

Therefore we have just proved that the system (4.27)&(4.28) is elliptic with the chosen weights in $\Omega \cap \mathcal{U}$, for some small neighborhood $\mathcal{U}$ of the origin.

On the other side, we have that

$$\begin{array}{ll} B_{11}(y,\xi) = 0 & B_{12}(y,\xi) = 1 \\ B_{21}(y,\xi) = -i\xi_1 & B_{22}(y,\xi) = i\xi_2. \end{array}$$

Thus we can take the obvious weights $r_1 = -2$ and $r_2 = -1$, so that the order of B_{hj} is less than or equal to $r_h + t_j$. Then the corresponding "principal symbol" B'_{hj} of B_{hj} coincides with B_{hj}, for every $h, j = 1, 2$.

It is clear that $s_1 + t_1 + s_2 + t_2 = 2\mu$. Therefore, in order to show that the boundary conditions (4.29)&(4.30) are coercive for the system (4.27)&(4.28), we only have to check that the system of equations

$$\begin{array}{rcll} L'_{j1}(0,D)U + L'_{j2}(0,D)V & = & 0 & \text{in } \mathbb{R}^2_+,\ j=1,2. \\ B'_{j1}(0,D)U + B'_{j2}(0,D)V & = & 0 & \text{on } y_2 = 0,\ j=1,2 \end{array}$$

admits no nontrivial bounded exponential solutions of the form:

$$U(y) = e^{i\xi y_1}\varphi(y_2), \qquad V(y) = e^{i\xi y_1}\varphi(y_2), \tag{4.31}$$

with $\xi \in \mathbb{R} \setminus \{0\}$.

In fact, the explicit expression of the above system is:

$$\begin{aligned} U_{;11} + \frac{1+\psi_{;1}^2}{\psi_{;2}^2} U_{;22} - 2\frac{\psi_{;1}}{\psi_{;2}} U_{;12} &= 0 \qquad \text{in } \mathbb{R}^2_+ \\ V_{;11} + \frac{1+\psi_{;1}^2}{\psi_{;2}^2} V_{;22} - 2\frac{\psi_{;1}}{\psi_{;2}} V_{;12} &= 0 \qquad \text{in } \mathbb{R}^2_+ \\ V(y_1,0) &= 0 \\ U_{;1}(y_1,0) - V_{;2}(y_1,0) &= 0, \end{aligned}$$

where all the partial derivatives of ψ are evaluated at the origin.

If the functions given by (4.31) are solutions to that system then both φ and φ satisfy the homogeneous second order linear differential equation

$$\frac{1+\psi_{;1}^2}{\psi_{;2}^2} g''(t) - 2i\frac{\psi_{;1}}{\psi_{;2}}\xi\, g'(t) - \xi^2 g(t) = 0,$$

and the boundary conditions $\varphi(0) = 0$ and $\varphi'(0) = i\xi\,\varphi(0)$.

The general form of the bounded solutions to the above ordinary differential equation is $g(t) = c \cdot e^{at}$, c being an arbitrary constant and

$$a = \left(-\frac{|\xi|}{|\psi_{;2}|} + i\frac{\psi_{;1}}{\psi_{;2}}\xi\right)\frac{\psi_{;2}^2}{1+\psi_{;1}^2}.$$

Thus the boundary conditions on φ and φ imply both functions must vanish identically.

Therefore we conclude that we may apply the regularity theorem [KS, Thm. VI.3.3] to the solutions ψ and W to our system (4.27)-(4.30) to obtain that they must be real analytic in some neighborhood of the origin. In particular, v is also real analytic in some small neighborhood of z^0. And, since the equation $v = 0$ describes Γ in that neighborhood, we get that, for every $z^0 \in \Gamma$, Γ admits a real analytic parameterization in some neighborhood of z^0. So we conclude that Γ is a real analytic curve, which is a contradiction to our hypothesis. Hence Theorem 2 has been proved.

5. Proof of the general mean-value theorem

The proof of Theorem 7 is carried out along the same lines as that of Theorem 1. For this reason, we will follow closely that proof, but without making explicit reference to it every time.

We only want to emphasize that the main difference between the proofs of the two theorems, loosely speaking, lies in the spherical Fourier transforms involved, and the discussion of their common zeroes.

5.1 First we want to write condition (1.8) in terms of integrals over a circle centered at the origin.

We know that the Euclidean disk $D = D(c, R)$ coincides with the pseudohyperbolic disk $\Delta(c_0, r)$ whose center $c_0 \in \mathbb{D}$ and radius $0 < r < 1$ are given by (2.4). Thus the automorphism σ_0, defined by (3.1), maps ∂D onto $\partial D(0, r)$, and by making the change of variable $r\eta = \sigma_0(\zeta)$ we obtain the following identity:

$$(5.1)\quad \int_{\partial D} g(\zeta)\,\frac{|d\zeta|}{2\pi R} = \int_{\partial\mathbb{D}} \mathcal{P}_{\mathbb{D}}(-rc_0, \eta)\,(g \circ \sigma_0^{-1})(r\eta)\,\frac{|d\eta|}{2\pi}, \text{ for every } g \in C(\mathbb{D}).$$

If we apply the above formula to the function $g(\zeta) = \mathcal{P}_D(a, \zeta)\,(f \circ \sigma \circ \sigma_0)(\zeta)$, we obtain that (1.8) is equivalent to

$$\int_{\partial\mathbb{D}} \mathcal{P}_{\mathbb{D}}(-rc_0, \zeta)\,\mathcal{P}_D(a, \sigma_0^{-1}(r\zeta))\,(f \circ \sigma)(r\zeta)\,\frac{|d\zeta|}{2\pi} = (f \circ \sigma)(a_0), \text{ for every } \sigma \in \mathcal{M},$$

where $a_0 = \sigma_0(a) \in D(0, r)$. It is clear that the above identity can be written as the convolution equation (3.3) in $\mathcal{M}$, where now $T = T_{\partial D, a}$ is the compactly supported Radon measure on $\mathbb{D}$ given by:

$$T\varphi = \int_{\partial\mathbb{D}} \mathcal{P}_{\mathbb{D}}(-rc_0, \zeta)\,\mathcal{P}_D(a, \sigma_0^{-1}(r\zeta))\,\varphi(r\zeta)\,\frac{|d\zeta|}{2\pi} - \varphi(a_0) \qquad (\varphi \in C(\mathbb{D})).$$

Observe that $T\varphi = 0$ for every harmonic function φ on $\mathbb{D}$, and, in particular, T vanishes on the constant functions.

5.2 Let $\mathcal{I}$ be the closed (convolution) ideal in $\mathcal{E}_0'(G)$ generated by the distributions of the form $\tilde{\tilde{T}} * (D_{j,k}\delta_0)\check{}$, $j, k \geq 0$. Let $I = \mathcal{F}(\mathcal{I})$ be the corresponding closed (multiplication) ideal in $\mathbb{E}'$. We know that this closed ideal is generated by the functions h_j, f_j, $j \geq 0$, given by (3.13) and (3.14).

Now let us compute those generators. Taking into account (3.7) and (3.15) it is clear that

$$h_j = F^{(j)}\left(\frac{r^2}{r^2-1}\right)\left(\frac{r}{r^2-1}\right)^j I_j - F^{(j)}\left(\frac{|a_0|^2}{|a_0|^2-1}\right)\left(\frac{\bar{a}_0}{|a_0|^2-1}\right)^j,$$

where

$$I_j = \int_{\partial\mathbb{D}} \mathcal{P}_{\mathbb{D}}(-rc_0, \zeta)\,\mathcal{P}_D(a, \sigma_0^{-1}(r\zeta))\,\bar{\zeta}^j\,\frac{|d\zeta|}{2\pi}$$

$$= \int_{\partial D} \mathcal{P}_D(a,\zeta) \left(\frac{\overline{\sigma_0(\zeta)}}{r}\right)^j \frac{|d\zeta|}{2\pi r}$$

$$= \left(\frac{\overline{\sigma_0(a)}}{r}\right)^j = \left(\frac{\overline{a}_0}{r}\right)^j.$$

The preceding second and third identities follow from applying formula (5.1) to the function $g = (\overline{\sigma}_0/r)^j$ and from the harmonicity of that function on $\mathbb{D}$, respectively.

Therefore

$$h_j = (-\overline{a}_0)^j \left\{ F^{(j)}\left(\frac{r^2}{r^2-1}\right)\frac{1}{(1-r^2)^j} - F^{(j)}\left(\frac{|a_0|^2}{|a_0|^2-1}\right)\frac{1}{(1-|a_0|^2)^j} \right\}.$$

Similarly, using now (3.7) and (3.16), we obtain that

$$f_j = (-a_0)^j \left\{ F^{(j)}\left(\frac{r^2}{r^2-1}\right)\frac{1}{(1-r^2)^j} - F^{(j)}\left(\frac{|a_0|^2}{|a_0|^2-1}\right)\frac{1}{(1-|a_0|^2)^j} \right\}.$$

Hence I is generated by the functions

$$g_j = F^{(j)}\left(\frac{r^2}{r^2-1}\right)\frac{1}{(1-r^2)^j} - F^{(j)}\left(\frac{|a_0|^2}{|a_0|^2-1}\right)\frac{1}{(1-|a_0|^2)^j}, \qquad (j \geq 0),$$

since $a_0 \neq 0$, because, by hypothesis, $a \neq c_0$.

5.3 We know that, since T vanishes on the constant functions, $\pm i$ are common zeroes of all the functions in I. On the other hand, g_0 vanishes at $\pm i$ with multiplicity equal to 1. In fact, observe that

$$g_0(\lambda) = \left[F\left(\frac{x}{x-1}\right)\right]_{x=|a_0|^2}^{r^2}.$$

Then, taking into account the formula

$$F(a,b;c;z) = (1-z)^{-a} F\left(a, c-b; c; \frac{z}{z-1}\right) \qquad (\operatorname{Re} z < 1/2)$$

(see [E, p. 64 (22)]), we have, for $0 \leq x < 1$, that

$$F\left(\frac{x}{x-1}\right) = (1-x)^{\frac{1+i\lambda}{2}} F\left(\frac{1+i\lambda}{2}, \frac{1+i\lambda}{2}; 1; x\right)$$

$$= (1-x)^{\frac{1+i\lambda}{2}} + \left(\frac{1+i\lambda}{2}\right)^2 (1-x)^{\frac{1+i\lambda}{2}} \sum_{k=1}^{\infty} \left(\frac{3+i\lambda}{2}\right)^2_{k-1} \frac{x^k}{(k!)^2}.$$

So $g_0 = g_{01} + g_{02}$, where

$$g_{01}(\lambda) = \left[(1-x)^{\frac{1+i\lambda}{2}}\right]_{x=|a_0|^2}^{r^2},$$

and g_{02} vanishes at i with multiplicity greater or equal than 2. But

$$g_{01}'(i) = \frac{i}{2}\log\frac{1-r^2}{1-|a_0|^2} \neq 0,$$

since $|a_0| < r$. Therefore we conclude g_0 has a zero with multiplicity equal to 1 at i (and then also at $-i$, since g_0 is an even function). Here it is interesting to note that all the other generators g_j, $j \geq 2$, of I vanish at $\pm i$ with multiplicity greater than or equal to 2.

In order to finish the proof of Theorem 7 it is clear that we only have to show that $\pm i$ are the unique common zeroes of the functions g_j, $j \geq 0$.

By multiplying equation (3.21) by $(1-z)^{k-2}$ and putting $z = t/(t-1)$, $0 \leq t < 1$, we obtain, for every $k \geq 2$, that

$$(5.2)\quad \Phi(t) = \frac{t}{(1-t)^k}F^{(k)}\left(\frac{t}{t-1}\right) = \frac{(k-1)(1+t)}{(1-t)^{k-1}}F^{(k-1)}\left(\frac{t}{t-1}\right) - \frac{(2k-3)^2+\lambda^2}{4(1-t)^{k-2}}F^{(k-2)}\left(\frac{t}{t-1}\right).$$

Therefore we have that:

$$\begin{aligned} g_k(\lambda) &= \frac{1}{r^2|a_0|^2}\left(|a_0|^2\Phi(r^2) - r^2\Phi(|a_0|^2)\right) \\ &= \frac{(k-1)(1+|a_0|^2)}{|a_0|^2}g_{k-1}(\lambda) - \frac{(2k-3)^2+\lambda^2}{4|a_0|^2}g_{k-2}(\lambda) \\ &\quad + \frac{|a_0|^2-r^2}{r^2|a_0|^2(1-r^2)^{k-2}}G_k(\lambda), \end{aligned}$$

where

$$(5.3)\quad G_k = \frac{k-1}{1-r^2}F^{(k-1)}\left(\frac{r^2}{r^2-1}\right) - \frac{(2k-3)^2+\lambda^2}{4}F^{(k-2)}\left(\frac{r^2}{r^2-1}\right).$$

Since $|a_0| < r$ it follows that every common zero $\lambda_0 \in \mathbb{C}$ of the functions g_k, $k \geq 0$, must be also a common zero of the functions G_k, $k \geq 2$.

Taking into account (5.2) for $t = r^2$, we get that

$$G_k = \frac{r^2}{1-r^2}\left\{\frac{1}{1-r^2}F^{(k)}\left(\frac{r^2}{r^2-1}\right) - (k-1)F^{(k-1)}\left(\frac{r^2}{r^2-1}\right)\right\}.$$

Then the holomorphic function $F_0(z) = F(\frac{1+i\lambda_0}{2}, \frac{1-i\lambda_0}{2}; 1; z)$ in $\operatorname{Re} z < 0$ satisfies

$$F_0^{(k)}(x_0) = (k-1)(1-r^2)F_0^{(k-1)}(x_0), \qquad \text{for every } k \geq 2,$$

where $x_0 = \frac{r^2}{r^2-1}$. It follows by induction that

$$F_0^{(k)}(x_0) = (k-1)!(1-r^2)^{k-1}F_0'(x_0), \qquad \text{for every } k \geq 2. \tag{5.4}$$

If $F_0'(x_0) = 0$ the above formula shows that $F_0^{(k)}(x_0) = 0$, for every $k \geq 2$, and, by analytic continuation, F_0 is constantly equal to $F_0(0) = 1$, and therefore λ_0 must be equal to $\pm i$.

Assume now $F_0'(x_0) \neq 0$. By (5.3) with $k = 3$, we have that

$$F_0''(x_0) = \frac{9+\lambda_0^2}{8}(1-r^2)F_0'(x_0). \tag{5.5}$$

Then by comparing (5.4) and (5.5) we get that $\lambda_0^2 = -1$, i.e., $\lambda_0 = \pm i$. Hence the proof of Theorem 7 is complete.

6. Some related problems

The purpose of this section is to discuss some problems related to those we deal with in the preceding sections of this article. Roughly speaking, we are going to consider the versions of the preceding mean-value and general mean-value problems where the Möbius group $\mathcal{M}$ acts on the circle ∂D instead of on the function f.

6.1 Let $D = D(c, R) \subset\subset \mathbb{D}$ be fixed. Since the relatively compact Euclidean disks in $\mathbb{D}$ coincide with the pseudohyperbolic disks, we know that, for every $\sigma \in \mathcal{M}$, $\sigma(D)$ is a relatively compact Euclidean disk in $\mathbb{D}$, namely, $\sigma(D) = D(c_\sigma, R_\sigma) \subset\subset \mathbb{D}$. Thus, if f is an harmonic function on $\mathbb{D}$, it is clear that

$$\int_{\sigma(\partial D)} f(\zeta)\frac{|d\zeta|}{2\pi R_\sigma} = f(c_\sigma), \qquad \text{for every } \sigma \in \mathcal{M}. \tag{6.1}$$

The problem now is to study whether every continuous function f on $\mathbb{D}$ satisfying (6.1) is harmonic.

It is evident that condition (6.1) can not be rewritten as a convolution equation on $\mathcal{M}$. So the methods we used in this paper are not applicable to this situation. Note that the corresponding version of the Morera problem is just equivalent to the invariant version we discuss in 1.2 (see Theorem 3).

We can only supply a partial solution to the mean-value problem we have just stated:

THEOREM 8: *If f is a bounded continuous function on $\mathbb{D}$ which verifies* (6.1) *then f is harmonic on $\mathbb{D}$.*

This result is just a particular case of the following theorem of Heath (which he proved using probabilistic methods):

THEOREM 9 ([Hea, Thm. 2]): *Let f be a bounded continuous function on $\mathbb{D}$. Let $\delta\colon \mathbb{D} \to \mathbb{R}$ be a function such that $0 < \delta(z) < 1 - |z|$, for every $z \in \mathbb{D}$, i.e., δ is a radius function.*

Assume that the radius function δ satisfies

$$|\delta(z) - \delta(w)| \leq k\,|z - w|, \qquad \text{for every } z, w \in \mathbb{D}, \tag{6.2}$$

for some constant $0 < k < 1$.

Suppose f possesses the restricted mean-value property on circles with respect to the radius function δ, i.e.,

$$f(z) = \int_{\partial D(z,\delta(z))} f(\zeta)\,\frac{|d\zeta|}{2\pi\delta(z)}, \qquad \text{for every } z \in \mathbb{D},$$

Then f is harmonic on $\mathbb{D}$.

(Theorem 9 is the unit disk version of the original Heath result, which holds on arbitrary proper open subsets of $\mathbb{R}^n$, $n \geq 2$.)

6.2 We are going to show how Theorem 8 follows from Theorem 9.

Observe that if c_0 and r are the pseudohyperbolic center and radius of $D = D(c, R)$, respectively, and $\sigma \in \mathcal{M}$, then $\sigma(D) = \Delta(\sigma(c_0), r)$, and so

$$c_\sigma = \frac{1 - r^2}{1 - r^2|\sigma(c_0)|^2}\,\sigma(c_0) \qquad \text{and} \qquad R_\sigma = \frac{1 - |\sigma(c_0)|^2}{1 - r^2|\sigma(c_0)|^2}\,r \qquad \text{(see (2.4))}.$$

Since the function

$$\varphi(x) = \frac{1 - r^2}{1 - r^2x^2}\,x$$

maps the interval $[0, 1)$ onto itself, it is now clear that for every $z \in \mathbb{D}$ we can select some $\sigma = \sigma_z \in \mathcal{M}$ such that $c_\sigma = z$. Then if a continuous function f on $\mathbb{D}$ satisfies (6.1), it possesses the restricted mean-value property on circles with respect to the radius function

$$\delta(z) = \frac{1 - |\sigma_z(c_0)|^2}{1 - r^2|\sigma_z(c_0)|^2}\,r.$$

Therefore Theorem 8 will become a particular case of Theorem 9 once we check our radius function δ satisfies condition (6.2).

In fact, for $z, w \in \mathbb{D}$, $z \neq w$, putting $a = \sigma_z(c_0)$ and $b = \sigma_w(c_0)$, we have that

$$|\delta(z) - \delta(w)| = (1 - r^2) r \frac{\left||a|^2 - |b|^2\right|}{(1 - r^2|a|^2)(1 - r^2|b|^2)}$$

and

$$\begin{aligned} |z - w| &= (1 - r^2)\frac{|a(1 - r^2|b|^2) - b(1 - r^2|a|^2)|}{(1 - r^2|a|^2)(1 - r^2|b|^2)} \\ &\geq (1 - r^2)\frac{\left||a|(1 - r^2|b|^2) - |b|(1 - r^2|a|^2)\right|}{(1 - r^2|a|^2)(1 - r^2|b|^2)} \\ &= (1 - r^2)\frac{(1 + r^2|a| \cdot |b|)\left||a| - |b|\right|}{(1 - r^2|a|^2)(1 - r^2|b|^2)}. \end{aligned}$$

Thus

$$\frac{|\delta(z) - \delta(w)|}{|z - w|} \leq r\frac{|a| + |b|}{1 + r^2|a| \cdot |b|} \leq \frac{2r}{1 + r^2}.$$

The last inequality follows from the fact that the middle term of the preceding chain of inequalities is an increasing function on any of both variables, $|a|$ or $|b|$, separately. Hence our radius function satisfies (6.2) with $k = 2r/(1 + r^2)$, which obviously satisfies $0 < k < 1$, and the proof of Theorem 8 is complete.

6.3 Finally we would like to state a general mean-value problem of a similar nature to the preceding one. We are going to use the notation introduced in 1.4, together with the one we employed in 6.1.

Let $D = D(c, R) \subset\subset \mathbb{D}$ and $a \in D$ be fixed. Then the problem is to decide whether every continuous function f on $\mathbb{D}$ possesing the following general mean-value property has to be necessarily harmonic:

$$\text{(6.3)} \quad \int_{\sigma(\partial D)} \mathcal{P}_{\sigma(D)}(\sigma(a), \zeta) f(\zeta) \frac{|d\zeta|}{2\pi R_\sigma} = f(\sigma(a)), \qquad \text{for every } \sigma \in \mathcal{M}.$$

Obviously every harmonic function on $\mathbb{D}$ satisfies that property. On the other hand, it is clear that (6.3) cannot be written as a convolution equation in the group $\mathcal{M}$, so, once again, we can not use the techniques employed in the present work. Moreover, we ignore whether a "restricted" general mean-value theorem (playing the same role that Theorem 9 does for the preceding mean-value problem) holds. Thus we even do not know the solution to that problem in the particular case of bounded functions.

ACKNOWLEDGEMENT: A great part of this work was done while the second author was visiting the University of Maryland (Fall 1989–Spring 1990, February 1991), sponsored by NSF grants DMS-8703072 and DMS-9000619. He would like to thank the Mathematics Department for its hospitality, and the first author for the invitation and for creating a very friendly and stimulating mathematical atmosphere during his visit.

References

[BCPZ] C. Berenstein, D-C. Chang, D. Pascuas and L. Zalcman, *Variations on the theorem of Morera,* Contemporary Math. **137** (1992), 63–78.

[BG] C.A. Berenstein and R. Gay, *Le problème de Pompeiu local,* J. Analyse Math. **52** (1988), 133–166.

[BS] C.A. Berenstein and M. Shahshahani, *Harmonic analysis and the Pompeiu problem,* Amer. J. Math. **105** (1983), 1217–1229.

[BZ1] C.A. Berenstein and L. Zalcman, *Pompeiu's problem on spaces of constant curvature,* J. Analyse Math. **30** (1976), 113–130.

[BZ2] C.A. Berenstein and L. Zalcman, *Pompeiu's problem on symmetric spaces,* Comment. Math. Helvetici **55** (1980), 593–621.

[C] L. A. Caffarelli, *The regularity of free boundaries in higher dimensions,* Acta Math. **139** (1977), 155–184.

[CP] C. Cascante and D. Pascuas, *Holomorphy tests based on Cauchy's integral formula,* preprint, (1993).

[E] A. Erdélyi, W. Magnus, F. Oberhettinge and F.G. Tricomi, *Higher Transcendental Functions,* Vol. 1, McGraw-Hill, New York, 1953.

[F] G. B. Folland, *Introduction to Partial Differential Equations,* Mathematical Notes 17, Princeton University Press, Princeton, New Jersey, 1976.

[G] J.B. Garnett, *Bounded Analytic Functions,* Pure and Applied Mathematics 96, Academic Press, New York, 1981.

[GT] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order,* Grundlehren der mathematischen Wissenschaften 224, Springer-Verlag, Berlin, 1977.

[Hea] D. Heath, *Functions possessing restricted mean-value properties,* Proc. Amer. Math. Soc. **41** (1973), 588–595.

[Hel1] S. Helgason, *Differential Geometry, Lie Groups and Symmetric Spaces,* Pure and Applied Mathematics 80, Academic Press, San Diego, 1978.

[Hel2] S. Helgason, *Groups and Geometric Analysis*, Pure and Applied Mathematics 113, Academic Press, Orlando, Florida, 1984.

[Hel3] S. Helgason, *The surjectivity of invariant differential operators I*, Ann. Math. **98** (1973), 451–479.

[Hel4] S. Helgason, *A duality for symmetric spaces, with applications to group representations*, Adv. Math. **5** (1970), 1–154.

[Hol] A.S.B. Holland, *Introduction to the Theory of Entire Functions*, Pure and Applied Mathematics 56, Academic Press, New York, 1973.

[Hor] L. Hörmander, *The Analysis of Linear Partial Differential Operators I*, Grundlehren der mathematischen Wissenschaften 256, Springer-Verlag, Berlin, 1983.

[Ka] J.P. Kahane, *Lectures on Mean Periodic Functions*, Lectures on Mathematics and Physics 15, Tata Institute of Fundamental Research, Bombay, 1959.

[KS] D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities and Their Applications*, Pure and Applied Mathematics 88, Academic Press, New York, 1980.

[Ko] P. Koosis, *The Logarithmic Integral I*, Cambridge studies in advanced mathematics 12, Cambridge University Press, Cambridge, 1988.

[L] N. N. Lebedev, *Special Functions and their Applications*, Dover, New York, 1972.

[LS] B.M. Levitan and I.S. Sargsjan, *Introduction to Spectral Theory: Selfadjoint Ordinary Differential Operators*, Transl. Math. Monographs 39, Amer. Math. Soc., Providence, Rhode Island, 1975.

[T] A. Terras, *Harmonic Analysis on Symmetric Spaces and Applications*, Springer-Verlag, New York, 1985.

[Z] L. Zalcman, *Offbeat integral geometry*, Amer. Math. Month. **87** (1980), 161–175.